# 8 Probability

GO DIGITAL

WATCH

INFO

## NATIONAL GEOGRAPHIC EXPLORER
## Jeffrey Ian Rose

Dr. Jeffrey Ian Rose is a prehistoric archaeologist specializing in the Paleolithic and Neolithic periods of the Arabian Peninsula. His areas of interest include modern human origins, Neolithization, stone tool technology, human genetics, rock art, geoarchaeology, underwater archaeology, and comparative religions.

• When did the Paleolithic and Neolithic periods occur?

• Which modern countries are in the Arabian Peninsula?

• Describe different types of stone tools that have been discovered from the Neolithic period. Are these different from the stone tools that have been discovered from the Paleolithic period?

## STEM

Archaeologists use probability to help determine where to excavate. In the Performance Task, you will help a team of archaeologists choose between three potential excavation sites.

**Prehistoric Archaeology**

# Preparing for Chapter 8

| Chapter Learning Target | Understand probability. |
|---|---|
| Chapter Success Criteria | ◆ I can define theoretical and experimental probability. |
| | ◆ I can use two-way tables to find probabilities. |
| | ■ I can compare independent and dependent events. |
| | ■ I can construct and interpret probability and binomial distributions. |

◆ Surface
■ Deep

## Chapter Vocabulary

**Work with a partner.** Discuss each of the vocabulary terms.

outcome                                  experimental probability
event                                    independent events
theoretical probability                  dependent events
geometric probability                    two-way table

## Mathematical Practices

### Model with Mathematics

*Mathematically proficient students apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

**Work with a partner.** During an interview, an archaeologist is asked about the experimental probabilities of finding several different objects during past expeditions. The answers are recorded in the table.

| Object | Probability |
|---|---|
| tools | *"about nine times out of ten"* |
| pottery | *"about 70 percent of the time"* |
| religious objects | *"about three tenths of the time"* |
| bones | *"one hundred percent"* |
| intact human skeleton | *"less than one percent"* |
| walls or other parts of structures | *"about one out of four times"* |

1. Order the objects by probability from least to greatest. Explain your method.

2. Compare the probability that the archaeologist finds religious objects with the probability that the archaeologist does *not* find pottery. Explain your reasoning.

3. Describe the likelihood of each event in the table.

# 8 Prepare WITH CalcChat®

## Using the Percent Proportion

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**Example 1** **What percent of 12 is 9?**

$$\frac{a}{w} = \frac{p}{100}$$      Write the percent proportion.

$$\frac{9}{12} = \frac{p}{100}$$      Substitute 9 for *a* and 12 for *w*.

$$100 \cdot \frac{9}{12} = 100 \cdot \frac{p}{100}$$      Multiplication Property of Equality

$$75 = p$$      Simplify.

▶ So, 9 is 75% of 12.

**Write and solve a proportion to answer the question.**

**1.** What percent of 30 is 6?     **2.** What number is 68% of 25?     **3.** 34.4 is what percent of 86?
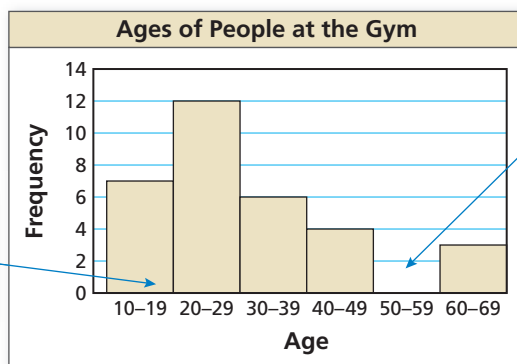
## Making a Histogram

WATCH

**Example 2** **The frequency table shows the ages of people at a gym. Display the data in a histogram.**

| Age | Frequency |
|-----|-----------|
| 10–19 | 7 |
| 20–29 | 12 |
| 30–39 | 6 |
| 40–49 | 4 |
| 50–59 | 0 |
| 60–69 | 3 |

**Step 1** Draw and label the axes.

**Step 2** Draw a bar to represent the frequency of each interval.



Ages of People at the Gym

There is no space between the bars of a histogram.

Include any interval with a frequency of 0. The bar height is 0.

**4.** Display the data in a histogram.

| Movies Watched per Week | | | |
|-----|-----|-----|-----|
| **Movies** | 0–1 | 2–3 | 4–5 |
| **Frequency** | 35 | 11 | 6 |

**5.** **MP** **REASONING** A sofa and an arm chair are the same price at a furniture store. The sofa is 20% off. The arm chair is 10% off, and a coupon applies an additional 10% off the discounted price of the chair. Are the items equally priced after the discounts are applied? Explain.

**Learning Target**   Find sample spaces and probabilities of events.

**Success Criteria**
- I can list the possible outcomes in a sample space.
- I can find theoretical probabilities.
- I can find experimental probabilities.

## EXPLORE IT! Finding Sample Spaces and Describing Events

**Work with a partner.**

**a.** Describe the set of all possible outcomes for each experiment.

   **i.** Three coins are flipped.      **ii.** One six-sided die is rolled.

   **iii.** Two six-sided dice are rolled.

**Math Practice**

**Understand Mathematical Terms**
Can the likelihood of an event be impossible? Can the likelihood of an event be certain?

**b.** Use your results in part (a) to describe the likelihood that the given event will occur. Explain your reasoning.

   **i.** You flip three tails.      **ii.** You roll an odd number.

   **iii.** You roll a sum greater than 3.

**c.** Use your results in part (a) to determine which event is more likely to occur. Explain your reasoning.

   **i.** **Event A:** flip exactly two heads  **ii.** **Event A:** roll an even number
       **Event B:** flip three heads            **Event B:** roll a number less than 3

   **iii.** **Event A:** roll "doubles"
       **Event B:** roll a sum less than 6

      **d.** Describe a real-life situation where it is important to know the likelihood of an event.
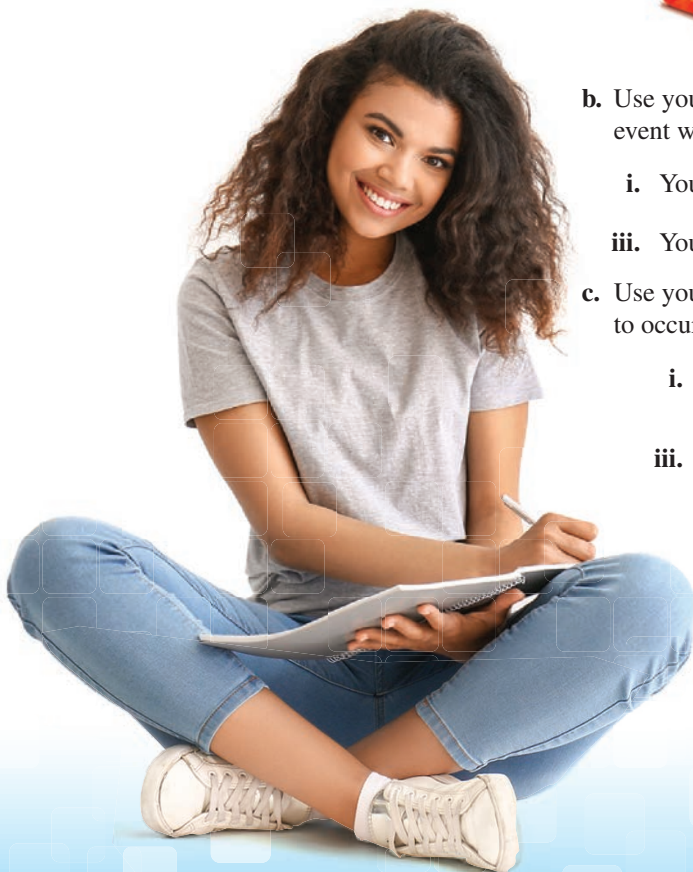
## Vocabulary

probability experiment, *p. 404*
outcome, *p. 404*
event, *p. 404*
sample space, *p. 404*
probability of an event, *p. 404*
theoretical probability, *p. 405*
geometric probability, *p. 406*
experimental probability, *p. 407*

## Sample Spaces

A **probability experiment** is an action, or trial, that has varying results. The possible results of a probability experiment are **outcomes**. A collection of one or more outcomes is an **event**. The set of all possible outcomes is called a **sample space**. Here are some examples.

**Probability experiment:** rolling a six-sided die

**Sample space:** 1, 2, 3, 4, 5, 6

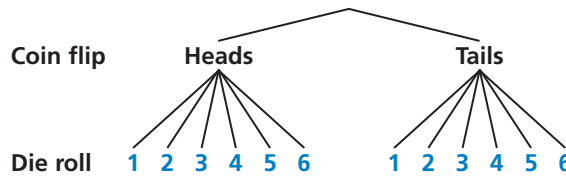**Event:** rolling an even number

**Outcome:** rolling a 4

**EXAMPLE 1** **Finding a Sample Space** WATCH

You flip a coin and roll a six-sided die. How many possible outcomes are in the sample space? List the possible outcomes.

### SOLUTION

Use a tree diagram to find the outcomes in the sample space.



▶ The sample space has 12 possible outcomes. They are listed below.

| | | | | | |
|---|---|---|---|---|---|
| Heads, 1 | Heads, 2 | Heads, 3 | Heads, 4 | Heads, 5 | Heads, 6 |
| Tails, 1 | Tails, 2 | Tails, 3 | Tails, 4 | Tails, 5 | Tails, 6 |

### Math Practice

**State the Meaning of Symbols**

Using H for "heads" and T for "tails," you can list the outcomes as shown below.

H1  H2  H3  H4  H5  H6
T1  T2  T3  T4  T5  T6

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**SELF-ASSESSMENT** [1] I do not understand. [2] I can do it with help. [3] I can do it on my own. [4] I can teach someone else.

**Find the number of possible outcomes in the sample space. Then list the possible outcomes.**

1. You flip two coins.

2. You flip two coins and roll a six-sided die.

3. **WRITING** Explain the difference between an outcome and an event.

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## Theoretical Probabilities

The **probability of an event** is a measure of the likelihood, or chance, that the event will occur. Probability is a number from 0 to 1, including 0 and 1, and can be expressed as a decimal, fraction, or percent.
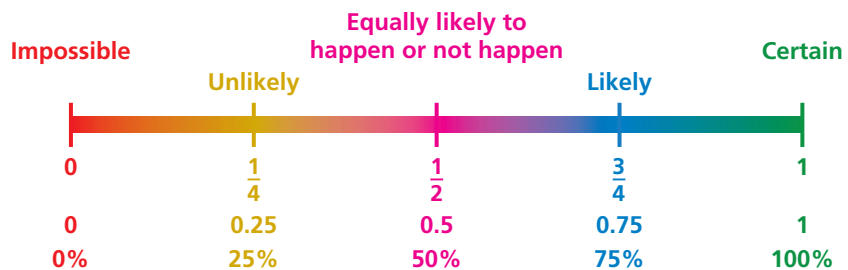
The outcomes for a specified event are called *favorable outcomes*. When all outcomes are equally likely, the **theoretical probability** of the event can be found using the following.

$$\text{Theoretical probability} = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

The probability of event $A$ is written as $P(A)$.

**EXAMPLE 2**  **Finding a Theoretical Probability**  ▶ WATCH

**Math Practice**

**Communicate Precisely**
Why is it more precise to use the phrase "exactly two answers" than the phrase "two answers?"

A student taking a quiz randomly guesses the answers to four true-false questions. What is the probability of the student guessing exactly two correct answers?

**SOLUTION**

**Step 1** Identify the sample space. Let C represent a correct answer and I represent an incorrect answer. The possible outcomes are shown in the table.

| Number correct | Outcome |
|---|---|
| 0 | IIII |
| 1 | CIII   ICII   IICI   IIIC |
| 2 | IICC   ICIC   ICCI   CIIC   CICI   CCII |
| 3 | ICCC   CICC   CCIC   CCCI |
| 4 | CCCC |

exactly two correct →

**Step 2** Identify the number of favorable outcomes and the total number of outcomes. There are 6 favorable outcomes with exactly two correct answers and the total number of outcomes is 16.

**Step 3** Find the probability of the student guessing exactly two correct answers. Because the student is randomly guessing, the outcomes should be equally likely. So, use the theoretical probability formula.

$$P(\text{exactly two correct answers}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

$$= \frac{6}{16}$$

$$= \frac{3}{8}$$

▶ The probability of the student guessing exactly two correct answers is $\frac{3}{8}$, or 37.5%.

The sum of the probabilities of all outcomes in a sample space is 1. So, when you know the probability of event $A$, you can find the probability of the *complement* of event $A$. The *complement* of event $A$ consists of all outcomes that are not in $A$ and is denoted by $\overline{A}$. The notation $\overline{A}$ is read as "$A$ bar." You can use the following formula to find $P(\overline{A})$.

**WORDS AND MATH**

*Complements* are parts that combine to create a whole. The probabilities of an event and its complement sum to 1.

💡 **KEY IDEA**

**Probability of the Complement of an Event**

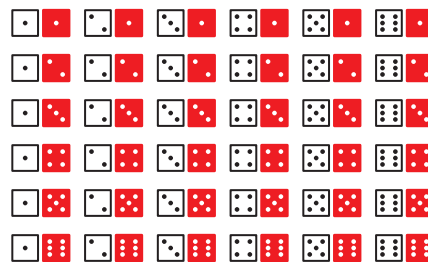The probability of the complement of event $A$ is

$$P(\overline{A}) = 1 - P(A).$$

**EXAMPLE 3** **Finding Probabilities of Complements**

When two six-sided dice are rolled, there are 36 possible outcomes, as shown. Find the probability of each event.

**a.** The sum is *not* 6.

**b.** The sum is less than or equal to 9.

**SOLUTION**

**a.** $P(\text{sum is not } 6) = 1 - P(\text{sum is } 6) = 1 - \frac{5}{36} = \frac{31}{36} \approx 0.861$

**b.** $P(\text{sum} \le 9) = 1 - P(\text{sum} > 9) = 1 - \frac{6}{36} = \frac{30}{36} = \frac{5}{6} \approx 0.833$

Some probabilities are found by calculating a ratio of two lengths, areas, or volumes. Such probabilities are called **geometric probabilities**.

**EXAMPLE 4** **Using Area to Find Probability**

You throw a dart at the board shown. Your dart is equally likely to hit any point inside the square board. Are you more likely to get 10 points or 0 points?

3 in.
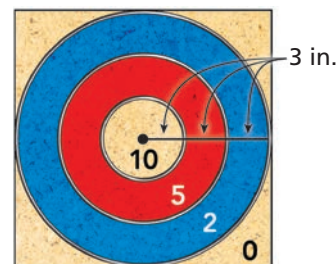
**SOLUTION**

The radius of the largest circle is 9 inches, so the side length of the board is 18 inches.

The probability of getting 10 points is

$$P(10 \text{ points}) = \frac{\text{Area of smallest circle}}{\text{Area of entire board}} = \frac{\pi \cdot 3^2}{18^2} = \frac{9\pi}{324} \approx 0.087.$$

> **REMEMBER**
> The area $A$ of a circle with radius $r$ is $A = \pi r^2$.

The probability of getting 0 points is

$$P(0 \text{ points}) = \frac{\text{Area outside largest circle}}{\text{Area of entire board}}$$

$$= \frac{18^2 - (\pi \cdot 9^2)}{18^2}$$

$$= \frac{324 - 81\pi}{324}$$

$$\approx 0.215.$$

▶ Because $0.215 > 0.087$, you are more likely to get 0 points.

**SELF-ASSESSMENT**  1 | I do not understand.  2 | I can do it with help.  3 | I can do it on my own.  4 | I can teach someone else.

**4.** You flip a coin and roll a six-sided die. What is the probability that the coin shows tails and the die shows 4?

**Use the information in Example 3 to find the probability of each event.**

**5.** The sum is *not* 11.

**6.** The sum is greater than 3.

**7.** In Example 4, are you more likely to get 5 points or 0 points?

**8.** In Example 4, are you more likely to score more than 2 points or exactly 2 points?

# Experimental Probabilities

An **experimental probability** is based on repeated *trials* of a probability experiment. The number of trials is the number of times the probability experiment is performed. Each trial in which a favorable outcome occurs is called a *success*. The experimental probability can be found using the following.

$$\text{Experimental probability} = \frac{\text{Number of successes}}{\text{Number of trials}}$$

## EXAMPLE 5  Finding an Experimental Probability

WATCH

**Spinner Results**

| red | green | blue | yellow |
|-----|-------|------|--------|
| 5   | 9     | 3    | 3      |

Each section of the spinner shown has the same area. The spinner is spun 20 times. The table shows the results. For which color is the experimental probability of stopping on the color the same as the theoretical probability?
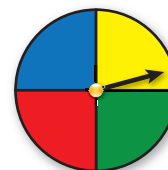
### SOLUTION

The theoretical probability of stopping on each of the four colors is $\frac{1}{4}$. Use the outcomes in the table to find the experimental probabilities.

$$P(\text{red}) = \frac{5}{20} = \frac{1}{4} \qquad P(\text{green}) = \frac{9}{20} \qquad P(\text{blue}) = \frac{3}{20} \qquad P(\text{yellow}) = \frac{3}{20}$$

▶ The experimental probability of stopping on red is the same as the theoretical probability.

## EXAMPLE 6  Modeling Real Life

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A research team finds that 368 out of 490 crustaceans have ingested plastic. The types of crustaceans that ingested plastic are shown. The team randomly selects a crustacean that ingested plastic to demonstrate their findings. What is the probability that they choose a crayfish?

**Crustaceans That Ingested Plastic**

Number of crustaceans

- Shrimp: 162
- Crayfish: 84
- Crab: 63
- Lobster: 59

### SOLUTION

The number of trials is the number of crustaceans that ingested plastic, 368. A success is a crustacean that ingested plastic is a crayfish. From the figure, there are 84 crayfish.
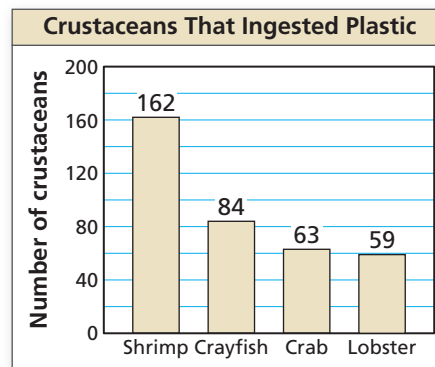
$$P(\text{crustacean that ingested plastic is a crayfish}) = \frac{84}{368} = \frac{21}{92} \approx 0.228$$

▶ The probability that they choose a crayfish is about 23%.

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**SELF-ASSESSMENT**  `1` I do not understand.  `2` I can do it with help.  `3` I can do it on my own.  `4` I can teach someone else.

9. In Example 5, for which color is the experimental probability of stopping on the color greater than the theoretical probability?

10. **WHAT IF?** In Example 6, what is the probability that they randomly select a lobster?

**In Exercises 1–4, find the number of possible outcomes in the sample space. Then list the possible outcomes.**
▶ *Example 1*

1. You flip a coin and draw a marble at random from a bag containing two purple marbles and one white marble.

2. You flip four coins.

3. You randomly choose a letter from A to F and a whole number from 1 to 3.

4. You draw two marbles without replacement from a bag containing three green marbles and three black marbles.

5. **FINDING A THEORETICAL PROBABILITY** A game show airs five days per week. Each day, a prize is randomly placed behind one of two doors. The contestant wins the prize by selecting the correct door. What is the probability that exactly two of the five contestants win a prize during a week?
▶ *Example 2*

6. **FINDING A THEORETICAL PROBABILITY** Your friend has two standard decks of 52 playing cards and asks you to randomly draw one card from each deck. What is the probability that you will draw two spades?

7. **FINDING PROBABILITIES OF COMPLEMENTS** When two six-sided dice are rolled, there are 36 possible outcomes. Find the probability that (a) the sum is *not* 4 and (b) the sum is greater than 5. ▶ *Example 3*

8. **FINDING PROBABILITIES OF COMPLEMENTS** The age distribution of guests at a cultural festival is shown. Find the probability that (a) a person chosen at random is at least 15 years old and (b) a person chosen at random is *not* 25 to 44 years old.

**Age Distribution**

- 25–34: 13%
- 35–44: 13%
- 15–24: 14%
- 45–54: 15%
- 5–14: 13%
- 55–64: 12%
- Under 5: 7%
- 65+: 13%
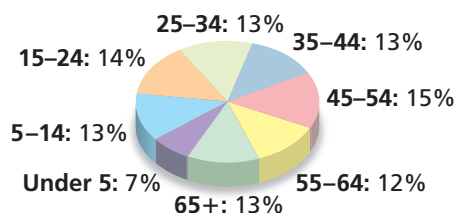
9. **ERROR ANALYSIS** A student randomly guesses the answers to two true-false questions. Describe and correct the error in finding the probability of the student guessing both answers correctly.

> ✗ The student can either guess two incorrect answers, two correct answers, or one of each. So the probability of guessing both answers correctly is $\frac{1}{3}$.

10. **ERROR ANALYSIS** A student randomly draws a whole number between 1 and 30. Describe and correct the error in finding the probability that the number drawn is greater than 4.

> ✗ $P(\text{number} > 4) = 1 - P(\text{number} < 4)$
> $= 1 - \frac{3}{30}$
> $= \frac{27}{30}$
> $= \frac{9}{10}$

11. **FINDING A GEOMETRIC PROBABILITY** You throw a dart at the board shown. Your dart is equally likely to hit any point inside the square board. What is the probability your dart lands in the yellow region?
▶ *Example 4*

18 in.
6 in.
18 in.
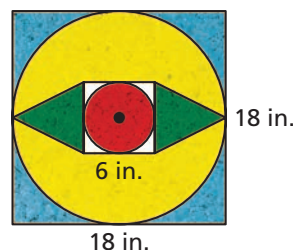
12. **FINDING A GEOMETRIC PROBABILITY** A student loses his earbuds while walking home from school. The earbuds are equally likely to be at any point along the path shown. What is the probability that the earbuds are on Cherry Street?

Park St.
0.4 mi
Pine St.
0.1 mi
School
Cherry St.
0.2 mi
Home
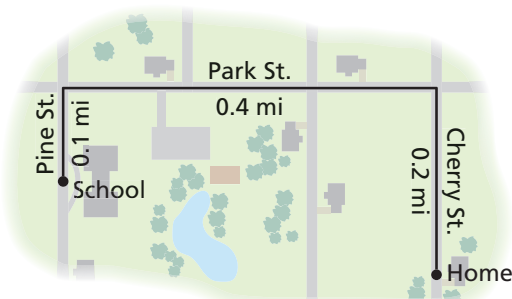
**13.** **DRAWING CONCLUSIONS** You roll a six-sided die 60 times. The table shows the results. For which number is the experimental probability of rolling the number the same as the theoretical probability? ▶ *Example 5*

| **Six-sided Die Results** | | | | | |
|---|---|---|---|---|---|
| • | •• | •• | :: | :: | :: |
| 11 | 14 | 7 | 10 | 6 | 12 |

**14.** **DRAWING CONCLUSIONS** A bag contains 5 marbles that are each a different color. A marble is drawn, its color is recorded, and then the marble is placed back in the bag. The table shows the results after 30 draws. For which marble(s) is the experimental probability of drawing the marble greater than the theoretical probability?

| **Drawing Results** | | | | |
|---|---|---|---|---|
| white | black | red | green | blue |
| 5 | 6 | 8 | 2 | 9 |

**15.** **MODELING REAL LIFE** An archaeologist uncovers 26 artifacts from a site. The types of artifacts are shown. An artifact is randomly selected for display. What is the probability that a piece of pottery is selected? ▶ *Example 6*



Archeological Artifacts
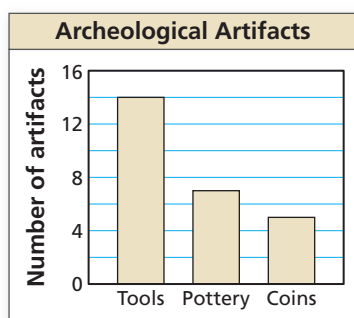
**16.** **MODELING REAL LIFE** A survey of 140 teenagers asked what type of food they like best. The results are shown. What is the probability that a randomly selected teenager from the survey likes Mexican food best?
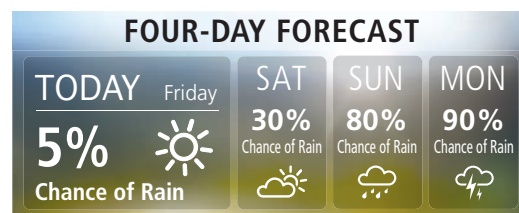


Survey Results

American — 31
Italian — 39
Mexican — 24
Chinese — 22
Japanese — 10
Other — 14
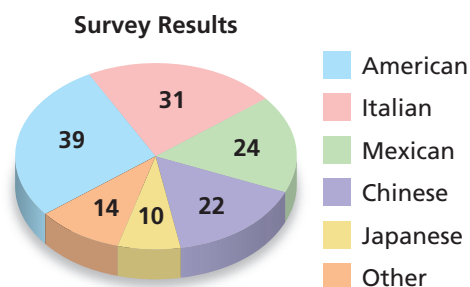
**17.** **MAKING AN ARGUMENT** You flip a coin three times. It lands on heads twice and on tails once. Your friend concludes that the theoretical probability of the coin landing heads up is $\frac{2}{3}$. Is your friend correct? Explain your reasoning.

**18.** **OPEN-ENDED** Describe a real-life event that has a probability of 0. Then describe a real-life event that has a probability of 1.

**19.** **ANALYZING RELATIONSHIPS** Refer to the board in Exercise 11. Order the likelihoods that the dart lands in the given region from least likely to most likely.

 **A.** green

 **B.** *not* blue

 **C.** red

 **D.** *not* yellow

**20.** **ANALYZING RELATIONSHIPS** Refer to the chart below. Order the following events from least likely to most likely.

**FOUR-DAY FORECAST**

| TODAY Friday | SAT | SUN | MON |
|---|---|---|---|
| 5% Chance of Rain | 30% Chance of Rain | 80% Chance of Rain | 90% Chance of Rain |

 **A.** It rains on Sunday.

 **B.** It does *not* rain on Saturday.

 **C.** It rains on Monday.

 **D.** It does *not* rain on Friday.

**21.** **MP** **USING TOOLS** Use the figure in Example 3.

 **a.** List the possible sums that result from rolling two six-sided dice.

 **b.** Find the theoretical probability of rolling each sum.

 **c.** The table shows a simulation of rolling two six-sided dice three times. Use a random number generator to simulate rolling two six-sided dice 50 times. Compare the experimental probabilities of rolling each possible sum with the theoretical probabilities.

| | A | B | C | |
|---|---|---|---|---|
| 1 | **First Die** | **Second Die** | **Sum** | |
| 2 | 4 | 6 | 10 | |
| 3 | 3 | 5 | 8 | |
| 4 | 1 | 6 | 7 | |
| 5 | | | | |

**22. HOW DO YOU SEE IT?**
Consider the graph of $f$ shown. What is the probability that the graph of $y = f(x) + c$ intersects the $x$-axis when $c$ is a randomly chosen integer from 1 to 6? Explain.
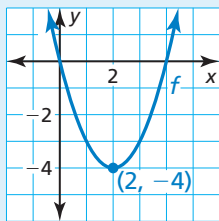


**23. CONNECTING CONCEPTS** A sphere fits inside a cube so that it touches each side, as shown. What is the probability a point chosen at random inside the cube is also inside the sphere?



**24. THOUGHT PROVOKING**
Describe a probability experiment that involves more than one action and has 48 possible outcomes in the sample space.

**25. DRAWING CONCLUSIONS**
A manufacturer tests 1200 computers and finds that 9 of them have defects. Predict the number of computers with defects in a shipment of 15,000 computers. Explain your reasoning.
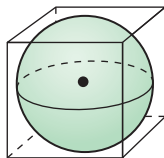
**26. DIG DEEPER** A test contains $n$ true-false questions. A student randomly guesses the answer to each question. Write an expression that gives the probability of correctly answering all $n$ questions.

**27. PERFORMANCE TASK** You are in charge of designing a game of chance for a fundraising event. You will charge a fee to play, and each winner will receive a cash prize. You expect about 200 people to play. Write a proposal in which you describe your game. Be sure to include how much you will charge to play, how much each winner will receive, the theoretical probability of winning, and how much you expect to raise (after prizes are deducted).

## REVIEW & REFRESH

**In Exercises 28 and 29, simplify the expression.**

**28.** $e^{\ln 9}$

**29.** $\log_3 81^{-2x}$

**In Exercises 30 and 31, describe the transformation of $f$ represented by $g$. Then graph each function.**

**30.** $f(x) = x^3$, $g(x) = -0.5x^3$

**31.** $f(x) = x^4$, $g(x) = (x + 1)^4 - 2$

**32.** The spinner is divided into sections with the same area. You spin the spinner 25 times. It stops on a multiple of 3 twenty times. Compare the experimental probability of spinning a multiple of 3 with the theoretical probability.
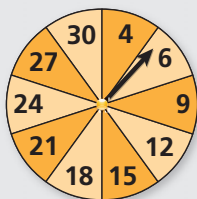


**33.** Solve $\dfrac{x}{x-3} - \dfrac{4}{x} = \dfrac{12}{x}$.

**34.** **MP REASONING** A polynomial function $f$ has rational coefficients and zeros of $2i$, $5 - i$, and $-3$. What is the least possible degree of $f$?

**In Exercises 35–40, perform the indicated operation.**

**35.** $\dfrac{4x^9y}{3x^3} \cdot \dfrac{2xy}{8y^2}$

**36.** $\dfrac{2y}{5x} \div \dfrac{y}{6x}$

**37.** $\dfrac{7}{x-2} + \dfrac{3x}{x+1}$

**38.** $\dfrac{4}{3x^2} - \dfrac{1}{x^2 + 5x}$

**39.** $\dfrac{x+8}{x^2 - x - 30} \cdot (x^2 - 7x + 6)$

**40.** $\dfrac{x^2 + 9x + 14}{x^2 - 9} \div \dfrac{x+7}{5x + 15}$

**41. MODELING REAL LIFE**
You want to order a pizza from one of two pizza shops. The table shows the total costs of a pizza with different numbers of toppings at Shop A. The total cost $y$ (in dollars) of a pizza with $x$ toppings at Shop B is represented by the equation $y = 1.75x + 11$. Which pizza shop charges less per topping? How many toppings must you order for the total costs to be the same?

| Shop A | |
|---|---|
| Number of toppings, $x$ | Total cost, $y$ |
| 1 | 13.50 |
| 2 | 15.00 |
| 3 | 16.50 |
| 4 | 18.00 |
| 5 | 19.50 |

**42.** Evaluate $625^{3/4}$.

**Learning Target**  Use two-way tables to represent data and find probabilities.

**Success Criteria**
- I can make two-way tables.
- I can find and interpret relative frequencies and conditional relative frequencies.
- I can use conditional relative frequencies to find probabilities.

## EXPLORE IT!   Finding Probabilities Using a Two-Way Table

### Math Practice

**Look for Structure**
What are some advantages and disadvantages of organizing data in a Venn diagram? in a table?

**Work with a partner.** A survey of 80 students at a high school asks whether they participate in outside of school activities and whether they participate in inside of school activities. The results are shown in the Venn diagram.

**Survey of 80 Students**

Participates in outside of school activity **32**

**16**

Participates in inside of school activity **23**

**9**
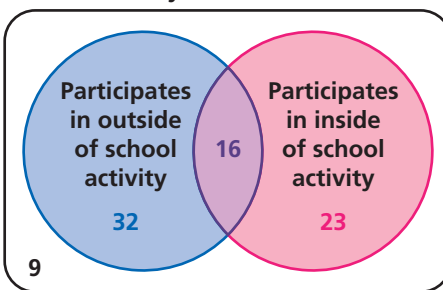
**a.** Show how you can represent the data in the Venn diagram using a single table.

| | | | |
|---|---|---|---|
| | | | |
| | | | |
| | | | |

**b.** One student is selected at random from the 80 students who took the survey. Find the probability that the student

   **i.** participates in an outside of school activity.

  **ii.** participates in an outside of school activity and participates in an inside of school activity.

 **iii.** participates in an outside of school activity and does not participate in an inside of school activity.

**c.** Conduct a survey of the students in your class. Choose two categories that are different from those given above. Then summarize the results in both a Venn diagram and a table similar to the one above. Discuss the results and find a probability using your table.

## Making Two-Way Tables

### Vocabulary

two-way table, *p. 412*
joint frequency, *p. 412*
marginal frequency, *p. 412*
joint relative frequency, *p. 413*
marginal relative frequency, *p. 413*
conditional relative frequency, *p. 414*

A **two-way table** is a frequency table that displays data collected from one source that belong to two different categories. One category of data is represented by rows and the other is represented by columns. For instance, the two-way table below shows the results of a survey that asked freshmen and sophomores whether they are attending a school concert.

Each entry in the table is called a **joint frequency**. The sums of the rows and columns are called **marginal frequencies**, which you will find in Example 1.

| | | Attendance | |
|---|---|---|---|
| | | Attending | Not Attending |
| Class | Freshman | 25 | 44 |
| | Sophomore | 80 | 32 |

joint frequency

---

### EXAMPLE 1  Making a Two-Way Table

In another survey similar to the one above, 106 juniors and 114 seniors respond. Of those, 42 juniors and 77 seniors plan on attending. Organize these results in a two-way table. Then find and interpret the marginal frequencies.

#### SOLUTION

**Step 1** Find the joint frequencies. Because 42 of the 106 juniors are attending, $106 - 42 = 64$ juniors are not attending. Because 77 of the 114 seniors are attending, $114 - 77 = 37$ seniors are not attending. Place each joint frequency in its corresponding cell.

**Step 2** Find the marginal frequencies. Create a new column and row for the sums. Then add the entries and interpret the results.

**Step 3** Find the sums of the marginal frequencies. Notice the sums $106 + 114 = 220$ and $119 + 101 = 220$ are equal. Place this value at the bottom right.

### READING

A two-way table is also called a *contingency table*, or a *two-way frequency table*.

| | | Attendance | | |
|---|---|---|---|---|
| | | Attending | Not Attending | Total |
| Class | Junior | 42 | 64 | 106 |
| | Senior | 77 | 37 | 114 |
| | Total | 119 | 101 | 220 |

106 juniors responded.
114 seniors responded.
220 students were surveyed.
119 students are attending.
101 students are not attending.

---

**SELF-ASSESSMENT**   **1** I do not understand.   **2** I can do it with help.   **3** I can do it on my own.   **4** I can teach someone else.

1. You randomly survey students about whether they are in favor of planting a community garden at school. Of 96 boys surveyed, 61 are in favor. Of 88 girls surveyed, 17 are against. Organize the results in a two-way table. Then find and interpret the marginal frequencies.

# Finding Relative and Conditional Relative Frequencies

You can display values in a two-way table as frequency counts (as in Example 1) or as *relative frequencies*.

## KEY IDEA

**Relative Frequencies**

A **joint relative frequency** is the ratio of a joint frequency to the total number of values or observations.

A **marginal relative frequency** is the sum of the joint relative frequencies in a row or a column.

---

**EXAMPLE 2**  Finding Joint and Marginal Relative Frequencies  WATCH

Use the survey results in Example 1 to make a two-way table that shows the joint and marginal relative frequencies. Interpret one of the joint relative frequencies and one of the marginal relative frequencies.

### SOLUTION

To find the joint relative frequencies, divide each frequency by the total number of students in the survey. Then find the sum of each row and each column to find the marginal relative frequencies.

|  |  | Attendance | | |
|---|---|---|---|---|
|  |  | **Attending** | **Not Attending** | **Total** |
| **Class** | **Junior** | $\frac{42}{220} \approx 0.191$ | $\frac{64}{220} \approx 0.291$ | 0.482 |
|  | **Senior** | $\frac{77}{220} = 0.35$ | $\frac{37}{220} \approx 0.168$ | 0.518 |
|  | **Total** | 0.541 | 0.459 | 1 |

▶ The joint relative frequency 0.291 means that about 29.1% of the students in the survey are juniors who are not attending the concert. So, the probability that a randomly selected student from the survey is a junior who is not attending the concert is about 29.1%.

The marginal relative frequency 0.518 means that about 51.8% of the students in the survey are seniors. So, the probability that a randomly selected student from the survey is a senior is about 51.8%.

---

**SELF-ASSESSMENT**   1 | I do not understand.   2 | I can do it with help.   3 | I can do it on my own.   4 | I can teach someone else.

2. Use the survey results in Exercise 1 to make a two-way table that shows the joint and marginal relative frequencies. Interpret one of the joint relative frequencies and one of the marginal relative frequencies.

3. **WRITING** Explain the differences between joint relative frequencies and marginal relative frequencies.

## KEY IDEA

### Conditional Relative Frequencies

A **conditional relative frequency** is the ratio of a joint relative frequency to the marginal relative frequency. You can find a conditional relative frequency using a row total or a column total of a two-way table.

---

**EXAMPLE 3**  **Finding Conditional Relative Frequencies**  WATCH

Use the survey results in Example 1. Make a two-way table that shows the conditional relative frequencies based on (a) the row totals and (b) the column totals. Interpret one of the conditional relative frequencies in each table.

### SOLUTION

**a.** Use the marginal relative frequency of each *row* in Example 2 to calculate the conditional relative frequencies.

|  |  | Attendance | |
|---|---|---|---|
|  |  | **Attending** | **Not Attending** |
| **Class** | **Junior** | $\dfrac{0.191}{0.482} \approx 0.396$ | $\dfrac{0.291}{0.482} \approx 0.604$ |
|  | **Senior** | $\dfrac{0.35}{0.518} \approx 0.676$ | $\dfrac{0.168}{0.518} \approx 0.324$ |

▶ The conditional relative frequency 0.604 means that about 60.4% of the juniors in the survey are not attending the concert. So, the probability that a randomly selected junior from the survey is not attending the concert is about 60.4%.

**b.** Use the marginal relative frequency of each *column* in Example 2 to calculate the conditional relative frequencies.

|  |  | Attendance | |
|---|---|---|---|
|  |  | **Attending** | **Not Attending** |
| **Class** | **Junior** | $\dfrac{0.191}{0.541} \approx 0.353$ | $\dfrac{0.291}{0.459} \approx 0.634$ |
|  | **Senior** | $\dfrac{0.35}{0.541} \approx 0.647$ | $\dfrac{0.168}{0.459} \approx 0.366$ |

▶ The conditional relative frequency 0.634 means that of the students in the survey who are not attending the concert, about 63.4% are juniors. So, given that a randomly selected student in the survey is not attending the concert, the probability that the student is a junior is about 63.4%.

---

**SELF-ASSESSMENT**  **1** I do not understand.  **2** I can do it with help.  **3** I can do it on my own.  **4** I can teach someone else.

4. Use the survey results in Exercise 1 to make a two-way table that shows the conditional relative frequencies based on the column totals. Interpret one of the conditional relative frequencies.

5. Use the relative frequencies in Example 2 or 3 to find the probability that a randomly selected student from the survey is a senior who is attending the concert.

**In Exercises 1 and 2, complete the two-way table.**

**1.**

| | | Preparation | | |
| --- | --- | --- | --- | --- |
| | | Studied | Did Not Study | Total |
| Grade | Pass | | 6 | |
| | Fail | | | 10 |
| | Total | 38 | | 50 |

**2.**

| | | Response | | |
| --- | --- | --- | --- | --- |
| | | Yes | No | Total |
| Role | Student | 56 | | |
| | Teacher | | 7 | 10 |
| | Total | | 49 | |

**3.** **MAKING TWO-WAY TABLES** You survey 171 males and 180 females at Grand Central Station in New York City. Of those, 132 males and 151 females wash their hands after using the public restrooms. Organize these results in a two-way table. Then find and interpret the marginal frequencies. ▶ *Example 1*

**4.** **MAKING TWO-WAY TABLES** A survey asks 80 seniors and 66 juniors whether they have a curfew. Of those, 59 seniors and 28 juniors say they have a curfew. Organize these results in a two-way table. Then find and interpret the marginal frequencies.

**5.** **MODELING REAL LIFE** Use the survey results from Exercise 3 to make a two-way table that shows the joint and marginal relative frequencies. Interpret one of the joint relative frequencies and one of the marginal relative frequencies. ▶ *Example 2*

**6.** **MODELING REAL LIFE** In a survey, 49 people received a flu vaccine before the flu season and 63 people did not receive the vaccine. Of those who received the flu vaccine, 16 people got the flu. Of those who did not receive the vaccine, 17 got the flu. Make a two-way table that shows the joint and marginal relative frequencies. Interpret one of the joint relative frequencies and one of the marginal relative frequencies.

**7.** **MODELING REAL LIFE** A survey finds that 73 people like horror movies and 87 people do not. Of those who like horror movies, 39 people have visited a haunted house. Of those who do not like horror movies, 42 people have visited a haunted house. Make a two-way table that shows the conditional relative frequencies based on (a) the horror movie totals and (b) the haunted house totals. Interpret one of the conditional relative frequencies in each table. ▶ *Example 3*

**8.** **MODELING REAL LIFE** Use the survey results from Exercise 6 to make a two-way table that shows the conditional relative frequencies based on (a) the flu vaccine totals and (b) the flu totals. Interpret one of the conditional relative frequencies in each table.

**9.** **ERROR ANALYSIS** The table shows the conditional relative frequencies for the results of a survey based on the row totals. Describe and correct the error in interpreting the data.

| | | Roller Coasters | |
| --- | --- | --- | --- |
| | | Like | Dislike |
| Grade | Upperclassmen | 0.729 | 0.271 |
| | Underclassmen | 0.707 | 0.293 |

✗ 0.271 means that about 27.1% of the students surveyed who dislike roller coasters are upperclassmen.

**10.** **MAKING AN ARGUMENT** The results of a survey are organized in a two-way table. Can you find the conditional relative frequencies without calculating the joint and marginal relative frequencies? Explain.

**11.** **MP** **PROBLEM SOLVING** Students in a history class are writing reports about either the Paleolithic era or the Neolithic era. Of the 14 boys and 17 girls in the class, 8 boys are writing their reports about the Paleolithic era and 9 girls are writing their reports about the Neolithic era. Find the probability that a randomly selected student from the class is a boy who is writing his report about the Neolithic era.

**12.** **MP** **PROBLEM SOLVING** Use the survey results in Exercise 4 to find the probability that a randomly selected senior from the survey has a curfew.

**13.** **MODELING REAL LIFE** A survey asks students whether they prefer math class or science class. Of the 150 male students surveyed, 62% prefer math class over science class. Of the female students surveyed, 74% prefer math class. Make a two-way table to show the number of students in each category if 350 students were surveyed.

**14.** **HOW DO YOU SEE IT?**
A survey asks teenagers and adults about whether their state should increase the minimum driving age. The two-way table shows the results.

|  |  | Age Group | | |
|---|---|---|---|---|
|  |  | Teenager | Adult | Total |
| Response | Yes | 45 | 880 | 925 |
|  | No | 456 | 120 | 576 |
|  | Total | 501 | 1000 | 1501 |

a. What does 120 represent?

b. What does 925 represent?

c. What does 1501 represent?

**15.** **MP** **STRUCTURE** A survey asks 481 students their gender and blood type. The two-way table shows the results.

|  |  | Blood Type | | | |
|---|---|---|---|---|---|
|  |  | A | B | AB | O |
| Gender | Male | 93 | 50 | 14 | 79 |
|  | Female | 89 | 50 | 30 | 76 |

a. Find the probability that a randomly selected student from the survey has blood type AB.

b. Find the probability that a randomly selected student from the survey with blood type O is male.

**16.** **THOUGHT PROVOKING**
Provide an example of a three-way table. Explain how you can find conditional relative frequencies for the data in the table. Then find and interpret one of the conditional relative frequencies.

**17.** **OPEN-ENDED** Create and conduct a survey in your class. Organize the results in a two-way table. Then create a two-way table that shows the joint and marginal relative frequencies. Use the relative frequencies to find a probability.

## REVIEW & REFRESH

WATCH

In Exercises 18 and 19, determine whether the inverse of $f$ is a function. Then find the inverse.

**18.** $f(x) = \dfrac{6}{13 - x}$

**19.** $f(x) = \dfrac{1}{x^2} + 5$

**20.** **MODELING REAL LIFE** A survey finds that 110 people ate breakfast and 30 people skipped breakfast. Of those who ate breakfast, 10 people felt tired. Of those who skipped breakfast, 10 people felt tired. Make a two-way table that shows the conditional relative frequencies based on the breakfast totals. Then interpret one of the conditional relative frequencies.

**21.** Simplify $\dfrac{\dfrac{2}{x-3}}{\dfrac{4}{x} + \dfrac{1}{x-3}}$.

**22.** When two six-sided dice are rolled, there are 36 possible outcomes. Find the probability that the sum is *not* 7.

**23.** Write a rule for $g$.
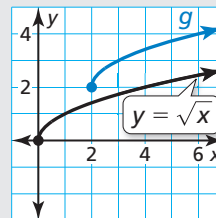


In Exercises 24 and 25, tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function.

**24.** $f(x) = (0.4)^x$

**25.** $y = 2e^{3x}$

**26.** Expand $\log_4 9x^3$.

**27.** Factor $a^3 - 2a^2 - 9a + 18$ completely.

In Exercises 28 and 29, solve the equation using any method. Explain your choice of method.

**28.** $2x^2 + 10 = 9x$

**29.** $3x^2 + 14x - 1 = 0$

# Learning Target

Find and use conditional probabilities.

**Success Criteria**
- I can explain the meaning of conditional probability.
- I can find conditional probabilities.
- I can make decisions using probabilities.

## EXPLORE IT! Finding Conditional Probabilities

**Work with a partner.** Six pieces of paper, numbered 1 through 6, are placed in a bag. You draw two pieces of paper one at a time without replacing the first.
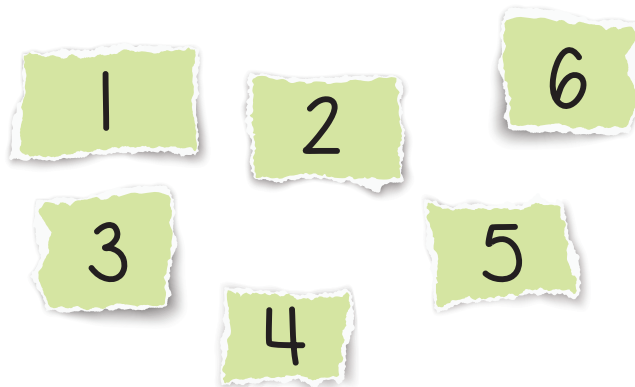
1   2   6

3   5

4

**a.** Use a tree diagram to find the outcomes in the sample space.

**b.** What is the probability that you draw two odd numbers?

**c.** When the first number you draw is odd, what is the probability that the second number you draw is also odd? Explain.

**d.** Compare and contrast the questions in parts (b) and (c).

**e.** The probability in part (c) is called a *conditional probability*. How would you define conditional probability?

**f.** The probability that event $B$ occurs given that event $A$ has occurred is called the conditional probability of $B$ given $A$ and is written as $P(B|A)$. The probability that both events $A$ and $B$ occur is written as $P(A \text{ and } B)$.

Find $P(B|A)$, $P(A \text{ and } B)$, and $P(A)$ for the following pair of events.

**Event $A$:** The first number is divisible by 3.
**Event $B$:** The second number is greater than 2.

**g.** Use your answers in part (f) to write a formula for $P(B|A)$ in terms of $P(A \text{ and } B)$ and $P(A)$.

### Math Practice

**Justify Conclusions**
Explain why your formula makes sense in part (g).

# Understanding Conditional Probability

The probability that event $B$ occurs given that event $A$ has occurred is called the **conditional probability** of $B$ given $A$ and is written as $P(B \mid A)$. You can use sample spaces and two-way tables to find conditional probabilities.

## EXAMPLE 1  Using a Sample Space to Find a Conditional Probability

WATCH

A family has two dogs and two cats. They randomly select a pet to get brushed and then randomly select a different pet to get a treat. Find the probability that they select a cat to get a treat given that they selected a dog to get brushed.

### SOLUTION

| Outcome | |
|---------|---------|
| $C_1D_1$ | $D_1C_1$ |
| $C_1D_2$ | $D_1C_2$ |
| $C_1C_2$ | $D_1D_2$ |
| $C_2C_1$ | $D_2D_1$ |
| $C_2D_1$ | $D_2C_1$ |
| $C_2D_2$ | $D_2C_2$ |

Let $C_1$ and $C_2$ represent the two cats, and $D_1$ and $D_2$ represent the two dogs. Use a table to list the outcomes in the sample space.

Use the sample space to find $P(\text{cat second} \mid \text{dog first})$. There are 6 outcomes for choosing a dog first. A cat is second in 4 of these 6 outcomes.

▶ So, $P(\text{cat second} \mid \text{dog first}) = \frac{4}{6} = \frac{2}{3} \approx 0.667$, or about 66.7%.

## EXAMPLE 2  Using a Two-way Table to Find Conditional Probabilities

WATCH

A quality-control inspector checks for defective parts. The two-way table shows the results. Find each probability.

**a.** $P(\text{pass} \mid \text{defective})$

**b.** $P(\text{fail} \mid \text{non-defective})$

| | | Result | |
|---|---|---|---|
| | | **Pass** | **Fail** |
| **Part Type** | **Defective** | 3 | 36 |
| | **Non-defective** | 450 | 11 |

### SOLUTION

**READING**

You can think of $P(B \mid A)$ as the fraction of $A$'s outcomes that are also in $B$. So, you can think of $P(\text{pass} \mid \text{defective})$ as the fraction of defective parts that pass the inspection.

**a.** Find the probability that a defective part passes.

$$P(\text{pass} \mid \text{defective}) = \frac{\text{Number of defective parts passed}}{\text{Total number of defective parts}}$$

$$= \frac{3}{3 + 36} = \frac{3}{39} = \frac{1}{13} \approx 0.077, \text{ or about } 7.7\%$$

**b.** Find the probability that a non-defective part fails.

$$P(\text{fail} \mid \text{non-defective}) = \frac{\text{Number of non-defective parts failed}}{\text{Total number of non-defective parts}}$$

$$= \frac{11}{450 + 11} = \frac{11}{461} \approx 0.024, \text{ or about } 2.4\%$$

---

**SELF-ASSESSMENT**   **1** I do not understand.   **2** I can do it with help.   **3** I can do it on my own.   **4** I can teach someone else.

1. In Example 1, what is the probability that they select a dog to get a treat given that they selected a dog to get brushed?

2. In Example 2, find (a) the probability that a non-defective part passes, and (b) the probability that a defective part fails.

---

## KEY IDEA

### Conditional Probability Formula

**READING**

The probability that both events A and B occur is written as P(A and B).

**Words** For two events $A$ and $B$, the conditional probability of the second event given the first event is the probability that both events occur divided by the probability of the first event.

**Symbols** $P(B|A) = \dfrac{P(A \text{ and } B)}{P(A)}$

**EXAMPLE 3** Using a Formula to Find a Conditional Probability

WATCH

Find the probability in Example 2(a) using the formula for conditional probability.

### SOLUTION

Find the joint and marginal relative frequencies.

| | | Result | | |
|---|---|---|---|---|
| | | **Pass** | **Fail** | **Total** |
| **Part Type** | **Defective** | $\dfrac{3}{500} = 0.006$ | $\dfrac{36}{500} = 0.072$ | 0.078 |
| | **Non-defective** | $\dfrac{450}{500} = 0.9$ | $\dfrac{11}{500} = 0.022$ | 0.922 |
| | **Total** | 0.906 | 0.094 | 1 |

Use the formula for conditional probability. Let event $A$ be "part is defective" and let event $B$ be "part passes."

$P(B|A) = \dfrac{P(A \text{ and } B)}{P(A)}$  Write formula for conditional probability.

$= \dfrac{0.006}{0.078}$  Substitute 0.006 for $P(A$ and $B)$ and 0.078 for $P(A)$.

$= \dfrac{6}{78}$  Rewrite fraction.

$= \dfrac{1}{13} \approx 0.077$  Simplify.

▶ So, the probability that a defective part passes is about 7.7%.

**SELF-ASSESSMENT**  | 1 | I do not understand. | 2 | I can do it with help. | 3 | I can do it on my own. | 4 | I can teach someone else. |

3. Find the probability in Example 2(b) using the formula for conditional probability.

4. You study survival statistics of adults from the Titanic shipwreck. The two-way table shows the joint and marginal relative frequencies. Find and compare $P(\text{female}|\text{did not survive})$ and $P(\text{female}|\text{survived})$.

| | | Result | | |
|---|---|---|---|---|
| | | **Survived** | **Did Not Survive** | **Total** |
| **Gender** | **Male** | 0.160 | 0.639 | 0.799 |
| | **Female** | 0.149 | 0.052 | 0.201 |
| | **Total** | 0.309 | 0.691 | 1 |

EXAMPLE 4   Finding Conditional Probabilities   ▶ WATCH   GO DIGITAL

At a school, 60% of students buy a school lunch, 18% of students buy a dessert, and 10% of students buy a lunch and a dessert.

**a.** What is the probability that a student who buys lunch also buys dessert?

**b.** What is the probability that a student who buys dessert also buys lunch?

## SOLUTION

Let event $A$ be "buys lunch" and let event $B$ be "buys dessert."

**a.** You are given $P(A) = 0.6$ and $P(A \text{ and } B) = 0.1$. Use the formula to find $P(B \mid A)$.

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} \qquad \text{Write formula for conditional probability.}$$

$$= \frac{0.1}{0.6} \qquad \text{Substitute 0.1 for } P(A \text{ and } B) \text{ and 0.6 for } P(A).$$

$$= \frac{1}{6} \qquad \text{Rewrite fraction.}$$

$$\approx 0.167 \qquad \text{Simplify.}$$

▶ So, the probability that a student who buys lunch also buys dessert is about 16.7%.

**b.** You are given $P(B) = 0.18$ and $P(A \text{ and } B) = 0.1$. Use the formula to find $P(A \mid B)$.

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \qquad \text{Write formula for conditional probability.}$$

$$= \frac{0.1}{0.18} \qquad \text{Substitute 0.1 for } P(A \text{ and } B) \text{ and 0.18 for } P(B).$$

$$= \frac{10}{18} \qquad \text{Rewrite fraction.}$$

$$= \frac{5}{9} \approx 0.556 \qquad \text{Simplify.}$$

▶ So, the probability that a student who buys dessert also buys lunch is about 55.6%.

**Math Practice**

**Use a Diagram**
Draw a Venn diagram that represents the values of $P(A)$, $P(B)$, $P(A \text{ and } B)$, and $P(\text{neither } A \text{ nor } B)$.

**SELF-ASSESSMENT**  | **1** I do not understand. | **2** I can do it with help. | **3** I can do it on my own. | **4** I can teach someone else.

5. At a coffee shop, 80% of customers order coffee, 15% of customers order coffee and a bagel, and 20% of customers order coffee and a sandwich.

   **a.** What is the probability that a customer who orders coffee also orders a bagel?

   **b.** What is the probability that a customer who orders coffee also orders a sandwich?

   **c.** What information do you need to find the probability that a customer who orders a bagel also orders coffee? Explain.

# Make Decisions Using Conditional Probabilities

**EXAMPLE 5** Using Conditional Probabilities to Make a Decision

WATCH | INFO

A jogger wants to burn a certain number of calories during her workout. She maps out three possible jogging routes. Before each workout, she randomly selects a route. Afterward, she uses a fitness tracker to determine whether she reaches her goal. The table shows her data. Which route should she use from now on?

| Route | Reaches Goal | Does Not Reach Goal |
|-------|--------------|---------------------|
| A | 卌 卌 I | 卌 I |
| B | 卌 卌 I | IIII |
| C | 卌 卌 II | 卌 I |

## SOLUTION

**Step 1** Use the data to make a two-way table that shows the joint and marginal relative frequencies. There are a total of 50 observations in the table.

**Step 2** Find the conditional probabilities by dividing each joint relative frequency in the "Reaches Goal" column by the marginal relative frequency in its corresponding row.

| | | Result | | |
|---|---|---|---|---|
| | | Reaches Goal | Does Not Reach Goal | Total |
| **Route** | **A** | 0.22 | 0.12 | 0.34 |
| | **B** | 0.22 | 0.08 | 0.30 |
| | **C** | 0.24 | 0.12 | 0.36 |
| | **Total** | 0.68 | 0.32 | 1 |

$$P(\text{reaches goal} \mid \text{Route A}) = \frac{P(\text{Route A and reaches goal})}{P(\text{Route A})} = \frac{0.22}{0.34} \approx 0.647$$

$$P(\text{reaches goal} \mid \text{Route B}) = \frac{P(\text{Route B and reaches goal})}{P(\text{Route B})} = \frac{0.22}{0.30} \approx 0.733$$

$$P(\text{reaches goal} \mid \text{Route C}) = \frac{P(\text{Route C and reaches goal})}{P(\text{Route C})} = \frac{0.24}{0.36} \approx 0.667$$

▶ Based on the sample, the probability that she reaches her goal is greatest when she uses Route B. So, she should use Route B from now on.

## SELF-ASSESSMENT

1 I do not understand. | 2 I can do it with help. | 3 I can do it on my own. | 4 I can teach someone else.

6. A manager is assessing three employees in order to offer one of them a promotion. Over a period of time, the manager records whether the employees meet or exceed expectations on their assigned tasks. The table shows the manager's results. Which employee should be offered the promotion? Explain.

| Employee | Exceed Expectations | Meet Expectations |
|----------|---------------------|-------------------|
| A | 卌 IIII | 卌 I |
| B | 卌 卌 II | 卌 III |
| C | 卌 卌 I | 卌 II |

1. **FINDING A PROBABILITY** A school lunch offers three different fruits and two different vegetables as side dishes. You are served two different side dishes at random. Find the probability that you are served a vegetable second given that you were served a fruit first. ▶ *Example 1*

2. **FINDING A PROBABILITY** A movie complex is showing three action films and three comedy films. You randomly select two different films to watch. Find the probability that you select a comedy to watch second given that you randomly selected a comedy to watch first.

3. **MODELING REAL LIFE** A teacher administers three different versions of a test to his students. The two-way table shows the results. Find each probability. ▶ *Example 2*

|  |  | Grade | |
|---|---|---|---|
|  |  | **Pass** | **Fail** |
| **Test** | **A** | 49 | 7 |
|  | **B** | 46 | 6 |
|  | **C** | 34 | 12 |

   a. $P(\text{pass} \mid \text{Test A})$     b. $P(\text{Test C} \mid \text{fail})$

4. **MODELING REAL LIFE** The two-way table shows the numbers of tropical cyclones that formed during the hurricane seasons over a 12-year period. Find each probability.
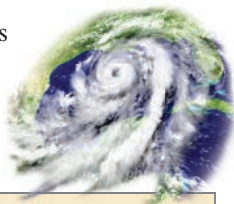
|  |  | Location | |
|---|---|---|---|
|  |  | **Northern Hemisphere** | **Southern Hemisphere** |
| **Type of Tropical Cyclone** | **Tropical depression** | 100 | 107 |
|  | **Tropical storm** | 342 | 487 |
|  | **Hurricane** | 379 | 525 |

   a. $P(\text{hurricane} \mid \text{Northern Hemisphere})$
   b. $P(\text{Southern Hemisphere} \mid \text{hurricane})$

5. **USING A FORMULA** Find the probability in Exercise 3(a) using the formula for conditional probability. ▶ *Example 3*

6. **USING A FORMULA** Find the probability in Exercise 4(a) using the formula for conditional probability.

**ERROR ANALYSIS** In Exercises 7 and 8, describe and correct the error in finding the given conditional probability.

|  |  | City | | | |
|---|---|---|---|---|---|
|  |  | **Tokyo** | **London** | **Washington, D.C.** | **Total** |
| **Satisfied Resident** | **Yes** | 0.049 | 0.136 | 0.171 | 0.356 |
|  | **No** | 0.341 | 0.112 | 0.191 | 0.644 |
|  | **Total** | 0.39 | 0.248 | 0.362 | 1 |

7. $P(\text{yes} \mid \text{Tokyo})$

   ✗ $$P(\text{yes} \mid \text{Tokyo}) = \frac{P(\text{Tokyo and yes})}{P(\text{Tokyo})}$$
   $$= \frac{0.049}{0.356} \approx 0.138$$

8. $P(\text{London} \mid \text{no})$

   ✗ $$P(\text{London} \mid \text{no}) = \frac{P(\text{no and London})}{P(\text{London})}$$
   $$= \frac{0.112}{0.248} \approx 0.452$$

9. **MODELING REAL LIFE** At a school, 43% of students attend the homecoming football game, 48% of students attend the homecoming dance, and 23% of students attend the game and the dance. ▶ *Example 4*

   a. What is the probability that a student who attends the football game also attends the dance?

   b. What is the probability that a student who attends the dance also attends the football game?

10. **MODELING REAL LIFE** At a gas station, 84% of customers buy gasoline, 9% of customers buy gasoline and a beverage, and 5% of customers buy gasoline and a snack.

   a. What is the probability that a customer who buys gasoline also buys a beverage?

   b. What is the probability that a customer who buys gasoline also buys a snack?

**11.** **MP** **PROBLEM SOLVING** You want to find the quickest route to school. You map out three routes. Before school, you randomly select a route and record whether you are late or on time. The table shows your findings. Assuming you leave at the same time each morning, which route should you use? Explain.
▶ *Example 5*

| Route | On Time | Late |
|---|---|---|
| A | 卅 II | IIII |
| B | 卅 卅 I | III |
| C | 卅 卅 II | IIII |

**12.** **MP** **PROBLEM SOLVING** A teacher is assessing three groups of students in order to award one group a prize. Over a period of time, the teacher records whether the groups meet or exceed expectations on their assigned tasks. The table shows the results. Which group should be awarded the prize? Explain.

| Group | Exceed Expectations | Meet Expectations |
|---|---|---|
| 1 | 卅 卅 II | IIII |
| 2 | 卅 III | 卅 |
| 3 | 卅 IIII | 卅 I |

**13.** **COLLEGE PREP** Let $P(A) = 0.6$, $P(B) = 0.8$, and $P(A \text{ and } B) = 0.24$. What is $P(A|B)$?

(A) 0.3
(B) 0.4
(C) 0.48
(D) 0.75

**14.** **MP** **STRUCTURE** Show that $P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$.

**15.** **MULTIPLE REPRESENTATIONS** The Venn diagram shows the results of a survey. Use the Venn diagram to construct a two-way table. Then use your table to answer each question.



Dog Owner — Cat Owner
57, 36, 25, 92
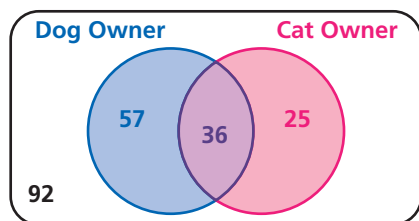
a. What is the probability that a randomly selected person from the survey does not own either pet?

b. What is the probability that a randomly selected person from the survey who owns a dog also owns a cat?

**16.** **HOW DO YOU SEE IT?**
You randomly select two marbles from a bag, one at a time. The sample space is shown, with each first selection on the left. What is the probability that you select a red marble second given that you randomly selected a blue marble first?
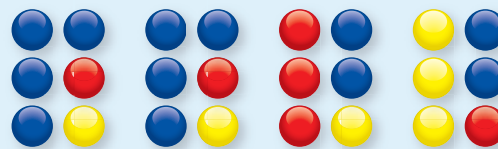


**17.** **MAKING AN ARGUMENT** Your friend uses the table to determine which dance routine meets the required time limit for a competition. Your friend decides that Routine B is the best option because it has the fewest tally marks in the "Does Not Meet Time Limit" column. Is your friend correct? Explain.

| Routine | Meets Time Limit | Does Not Meet Time Limit |
|---|---|---|
| A | 卅 | III |
| B | IIII | II |
| C | 卅 II | IIII |

**18.** **MP** **PROBLEM SOLVING** A pharmaceutical company conducts a voluntary study of 500 teenagers with acne to determine the effectiveness of a new medication. The two-way table shows the results. The company claims that the medicine is effective in 94% of cases. Do the data support this claim? Explain your reasoning.

| | | Took Medicine | |
|---|---|---|---|
| | | Yes | No |
| Acne Remaining | Yes | 98 | 134 |
| | No | 187 | 81 |

**19.** **CRITICAL THINKING** In a survey, 53% of respondents have a music streaming subscription, 68% have a video streaming subscription, and 47% of the respondents who have video streaming also have music streaming.

a. What is the probability that a person from the survey has both video and music streaming?

b. What is the probability that a person from the survey who has music streaming also has video streaming?
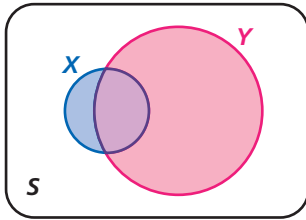
**20. THOUGHT PROVOKING**
Bayes' Theorem is given by

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}.$$

Use a two-way table to write an example of Bayes' Theorem.

**21. ABSTRACT REASONING** The Venn diagram represents the sample space $S$ for two events $X$ and $Y$. The area of each region is proportional to the number of outcomes within the region. Determine whether the statement $P(X|Y) > P(Y|X)$ is *true* or *false*. Explain.



**22.** DIG DEEPER A company creates a new recipe for a snack and tests it against its current recipe. The table shows the results.

|  |  | Recipe Preference | |
|---|---|---|---|
|  |  | Current | New |
| Current Consumer of Snack | Yes | 72 | 46 |
|  | No | 52 | 114 |

The company is deciding whether it should change the snack's recipe, and to whom the snack should be marketed. Use probability to explain the decisions the company should make when the total size of the snack's market is expected to (a) change very little, and (b) expand very rapidly.

## REVIEW & REFRESH

WATCH

**In Exercises 23 and 24, solve the equation.**

**23.** $\frac{9}{10}x = 0.18$

**24.** $\frac{1}{4}x + 0.5x = 1.5$

**In Exercises 25 and 26, identify the focus, directrix, and axis of symmetry of the parabola. Graph the equation.**

**25.** $x = \frac{1}{12}y^2$

**26.** $y = \frac{1}{4}(x + 3)^2 - 2$

**27.** Use the data to create a two-way table that shows the joint and marginal relative frequencies.

|  |  | Dominant Hand | | |
|---|---|---|---|---|
|  |  | Left | Right | Total |
| Gender | Female | 11 | 104 | 115 |
|  | Male | 24 | 92 | 116 |
|  | Total | 35 | 196 | 231 |

**28.** Solve $\dfrac{3}{x + 4} + \dfrac{2x^2}{x^2 - 16} = \dfrac{1}{x - 4}$.

**In Exercises 29 and 30, find $(f + g)(x)$ and $(f - g)(x)$ and state the domain of each. Then evaluate $f + g$ and $f - g$ for the given value of $x$.**

**29.** $f(x) = 6\sqrt[3]{4x}$, $g(x) = -12\sqrt[3]{4x}$; $x = -16$

**30.** $f(x) = 4x + 9x^2 - 2$, $g(x) = 10 - 3x^2$; $x = 1$

**In Exercises 31–34, find the product.**

**31.** $5x^2(2x^3 - x^2 + 13)$

**32.** $(2x + 7)(2x - 7)$

**33.** $(x^2 - 8x + 7)(x^2 + 2x - 3)$

**34.** $(x - 4)(2x + 3)(x + 1)$

**35.** You roll a six-sided die 30 times. A 5 is rolled 8 times. What is the theoretical probability of rolling a 5? What is the experimental probability of rolling a 5?

**36.** Find the zeros of $f(x) = 2x^3 + 9x^2 - 5x$. Then sketch a graph of the function.

**37. MODELING REAL LIFE** The two-way table shows the numbers of juniors at a school who have a gym membership and the numbers of juniors at a school who play a sport. Find each probability.

|  |  | Gym Membership | |
|---|---|---|---|
|  |  | Yes | No |
| Plays a Sport | Yes | 57 | 45 |
|  | No | 34 | 41 |

**a.** $P(\text{membership}|\text{sport})$

**b.** $P(\text{no sport}|\text{no membership})$

# Independent and Dependent Events

**Learning Target**
Understand and find probabilities of independent and dependent events.

**Success Criteria**
- I can explain how independent events and dependent events are different.
- I can determine whether events are independent.
- I can find probabilities of independent and dependent events.

## EXPLORE IT! Identifying Independent and Dependent Events

**Work with a partner.**

**a.** Two events are either *independent* or *dependent*. Which pair of events below are independent? Which are dependent? Explain your reasoning.

    **i.** You roll a six-sided die twice.

        **Event A:** The first number is even.

        **Event B:** The second number is a 6.

    **ii.** Six pieces of paper, numbered 1 through 6, are placed in a bag. Two pieces of paper are selected one at a time without replacement.

        **Event A:** The first number is even.

        **Event B:** The second number is a 6.

**b.** Complete the table for each set of events in part (a).

| Experiment | Rolling Die | Selecting Papers |
|---|---|---|
| $P(A)$ | | |
| $P(B)$ | | |
| $P(B \mid A)$ | | |
| $P(A \text{ and } B)$ | | |

**Math Practice**

**Interpret Results**
In the first experiment, $P(B \mid A) = P(B)$, but that is not true in the second experiment. Why? What does this imply about events $A$ and $B$?

**c.** Write a formula that relates $P(A \text{ and } B)$, $P(B \mid A)$, and $P(A)$.

$$P(A \text{ and } B) = \rule{3cm}{0.4pt}$$

# Determining Whether Events Are Independent

Two events are **independent events** when the occurrence of one event does not affect the occurrence of the other event.

## KEY IDEA

### Probability of Independent Events

**Words** Two events $A$ and $B$ are independent events if and only if the probability that both events occur is the product of the probabilities of the events.

**Symbols** $P(A \text{ and } B) = P(A) \cdot P(B)$

---

**EXAMPLE 1** Determining Whether Events Are Independent ▶ WATCH

A student taking a quiz randomly guesses the answers to four true-false questions. Use a sample space to determine whether guessing the correct answer to Question 1 and guessing the correct answer to Question 2 are independent events.

### SOLUTION

Use a table to list the outcomes in the sample space. Using the sample space:

| Outcomes | | | |
|------|------|------|------|
| IIII | IIIC | CIIC | CICC |
| CIII | IICC | CICI | CCIC |
| ICII | ICIC | CCII | CCCI |
| IICI | ICCI | ICCC | CCCC |

$P(\text{correct on Question 1}) = \frac{8}{16} = \frac{1}{2}$

$P(\text{correct on Question 2}) = \frac{8}{16} = \frac{1}{2}$

$P(\text{correct on Question 1 and correct on Question 2}) = \frac{4}{16} = \frac{1}{4}$

▶ Because $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, the events are independent.

---

**EXAMPLE 2** Determining Whether Events Are Independent ▶ WATCH

A group of four students includes one boy and three girls. The teacher randomly selects one of the students to be the speaker and a different student to be the recorder. Use a sample space to determine whether randomly selecting a girl first and randomly selecting a girl second are independent events.

### SOLUTION

Let B represent the boy. Let $G_1$, $G_2$, and $G_3$ represent the three girls. Use a table to list the outcomes in the sample space.

Using the sample space:

$P(\text{girl first}) = \frac{9}{12} = \frac{3}{4}$

$P(\text{girl second}) = \frac{9}{12} = \frac{3}{4}$

$P(\text{girl first and girl second}) = \frac{6}{12} = \frac{1}{2}$

▶ Because $\frac{3}{4} \cdot \frac{3}{4} \neq \frac{1}{2}$, the events are not independent.

| Number of girls | Outcome | |
|---|---|---|
| 1 | $G_1B$ | $BG_1$ |
| 1 | $G_2B$ | $BG_2$ |
| 1 | $G_3B$ | $BG_3$ |
| 2 | $G_1G_2$ | $G_2G_1$ |
| 2 | $G_1G_3$ | $G_3G_1$ |
| 2 | $G_2G_3$ | $G_3G_2$ |

You can also determine whether two events are independent using conditional probabilities.

## KEY IDEA

### Conditional Probability and Independent Events

**Words** When two events $A$ and $B$ are independent, the conditional probability of $A$ given $B$ is equal to the probability of $A$, and the conditional probability of $B$ given $A$ is equal to the probability of $B$.

**Symbols** $P(A|B) = P(A)$ and $P(B|A) = P(B)$

**EXAMPLE 3** Determining Whether Events Are Independent

WATCH

Use conditional probabilities to determine whether the events are independent in (a) Example 1 and (b) Example 2.

### SOLUTION

**a.** Determine whether $P(B|A) = P(B)$.

$$P(\text{correct on Question 2}\,|\,\text{correct on Question 1}) = \frac{4}{8}$$
$$= \frac{1}{2}$$
$$P(\text{correct on Question 2}) = \frac{8}{16}$$
$$= \frac{1}{2}$$

▶ This shows that being correct on the first question does not affect the probability of being correct on the second question. Because $P(B|A) = P(B)$, the events are independent.

**b.** Determine whether $P(B|A) = P(B)$.

$$P(\text{girl second}\,|\,\text{girl first}) = \frac{6}{9}$$
$$= \frac{2}{3}$$
$$P(\text{girl second}) = \frac{9}{12}$$
$$= \frac{3}{4}$$

▶ This shows that the first selection affects the outcome of the second selection. Because $P(B|A) \neq P(B)$, the events are not independent.

**SELF-ASSESSMENT**  1 I do not understand.  2 I can do it with help.  3 I can do it on my own.  4 I can teach someone else.

1. In Example 1, determine whether guessing Question 1 incorrectly and guessing Question 2 correctly are independent events.

2. In Example 2, determine whether randomly selecting a girl first and randomly selecting a boy second are independent events.

3. Five out of eight tiles in a bag have numbers on them. You randomly draw a tile, set it aside, and then randomly draw another tile. Use a conditional probability to determine whether selecting a numbered tile first and a numbered tile second are independent events.

| EXAMPLE 4 | Using a Two-Way Table to Determine Independence |

A satellite TV provider surveys customers in three cities. The survey asks whether they would recommend the TV provider to a friend. The results, given as joint relative frequencies, are shown in the two-way table. Determine whether recommending the provider to a friend and living in Long Beach are independent events.

| | | Location | | |
|---|---|---|---|---|
| | | Glendale | Santa Monica | Long Beach |
| Response | Yes | 0.29 | 0.27 | 0.32 |
| | No | 0.05 | 0.03 | 0.04 |

**SOLUTION**

Use the formula $P(B) = P(B|A)$ and compare $P(\text{Long Beach})$ and $P(\text{Long Beach}|\text{yes})$.

$$P(\text{Long Beach}) = 0.32 + 0.04$$
$$= 0.36$$

$$P(\text{Long Beach}|\text{yes}) = \frac{P(\text{yes and Long Beach})}{P(\text{yes})}$$
$$= \frac{0.32}{0.29 + 0.27 + 0.32}$$
$$\approx 0.36$$

▶ Because $P(\text{Long Beach}) \approx P(\text{Long Beach}|\text{yes})$, the two events are independent.

**SELF-ASSESSMENT** | **1** I do not understand. | **2** I can do it with help. | **3** I can do it on my own. | **4** I can teach someone else.

4. In Example 4, determine whether recommending the provider to a friend and living in Santa Monica are independent events. Explain your reasoning.

## Finding Probabilities of Events

In Example 1, it makes sense that the events are independent because the second guess should not be affected by the first guess. In Example 2, however, the selection of the second person *depends* on the selection of the first person because the same person cannot be selected twice. These events are *dependent*. Two events are **dependent events** when the occurrence of one event *does* affect the occurrence of the other event.

### KEY IDEA

**Probability of Dependent Events**

**Words** If two events $A$ and $B$ are dependent events, then the probability that both events occur is the product of the probability of the first event and the conditional probability of the second event given the first event.

**Symbols** $P(A \text{ and } B) = P(A) \cdot P(B|A)$

**EXAMPLE 5**  **Finding the Probability of Independent Events**

A spinner is divided into equal parts. Find the probability that you get a 5 on your first spin and a number greater than 3 on your second spin.

**SOLUTION**

Let event $A$ be "5 on first spin" and let event $B$ be "greater than 3 on second spin."

The events are independent because the outcome of your second spin is not affected by the outcome of your first spin. Find the probability of each event and then multiply the probabilities.

$P(A) = \frac{1}{8}$     1 of the 8 sections is a "5."

$P(B) = \frac{5}{8}$     5 of the 8 sections (4, 5, 6, 7, 8) are greater than 3.

$P(A \text{ and } B) = P(A) \cdot P(B) = \frac{1}{8} \cdot \frac{5}{8} = \frac{5}{64} \approx 0.078$

▶ So, the probability that you get a 5 on your first spin and a number greater than 3 on your second spin is about 7.8%.

**EXAMPLE 6**  **Finding the Probability of Dependent Events**

A bag contains twenty $1 bills and five $10 bills. You randomly draw a bill from the bag, set it aside, and then randomly draw another bill from the bag. Find the probability that both events $A$ and $B$ will occur.

**Event $A$:** The first bill is $10.

**Event $B$:** The second bill is $10.

**SOLUTION**

The events are dependent because there is one less bill in the bag on your second draw than on your first draw. Find $P(A)$ and $P(B|A)$. Then multiply the probabilities.

$P(A) = \frac{5}{25}$     5 of the 25 bills are $10 bills.

$P(B|A) = \frac{4}{24}$     When the first bill is $10, 4 of the remaining 24 bills are $10 bills.

$P(A \text{ and } B) = P(A) \cdot P(B|A) = \frac{5}{25} \cdot \frac{4}{24} = \frac{1}{5} \cdot \frac{1}{6} = \frac{1}{30} \approx 0.033.$

▶ So, the probability that you draw two $10 bills is about 3.3%.

**SELF-ASSESSMENT**  [1] I do not understand.  [2] I can do it with help.  [3] I can do it on my own.  [4] I can teach someone else.

5. In Example 5, what is the probability that you spin an even number and then an odd number?

6. In Example 6, what is the probability that both bills are $1 bills?

In Exercises 1–6, use a sample space to determine whether the events are independent. ▶ *Examples 1 and 2*

1. You play a game that involves spinning the spinner shown. Each section of the spinner has the same area. Determine whether randomly spinning blue and then green are independent events.
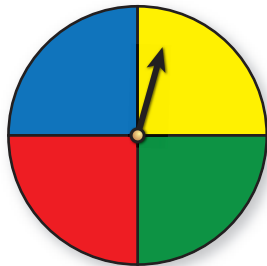
2. There are three green apples and one red apple in a bowl. You randomly select one apple to eat now and another apple to eat with lunch. Determine whether randomly selecting a green apple first and randomly selecting a green apple second are independent events.

3. A student is taking a multiple-choice quiz where each question has four choices. The student randomly guesses the answers to the three-question quiz. Determine whether guessing the correct answer to Question 1 and guessing the correct answer to Question 2 are independent events.

4. A bag contains four quarters and one nickel. You randomly select two coins. Determine whether randomly selecting a quarter first and randomly selecting a quarter second are independent events.

5. You randomly select two gift cards from a bag that contains three restaurant gift cards and two department store gift cards. Determine whether randomly selecting a restaurant gift card first and randomly selecting a department store gift card second are independent events.

6. You roll a six-sided die two times. Determine whether getting a 6 and then getting a 1 are independent events.

In Exercises 7–10, use a conditional probability to determine whether the events are independent. ▶ *Example 3*

7. Determine whether the events in Exercise 1 are independent.

8. Determine whether the events in Exercise 2 are independent.

9. You have six solid and three striped cell phone cases. You randomly select a case, set it aside, and then randomly select another case. Determine whether randomly selecting a striped case first and a striped case second are independent events.

10. You flip a coin and roll a six-sided die. Determine whether getting tails and getting a 4 are independent events.

11. **MODELING REAL LIFE** Three different local hospitals in New York surveyed their patients. The survey asked whether the patient's physician communicated efficiently. The results, given as joint relative frequencies, are shown in the two-way table. Determine whether being satisfied with the communication of the physician and living in Saratoga are independent events. ▶ *Example 4*

| | | Location | | |
|---|---|---|---|---|
| | | **Glens Falls** | **Saratoga** | **Albany** |
| **Response** | **Yes** | 0.123 | 0.289 | 0.338 |
| | **No** | 0.042 | 0.095 | 0.113 |

12. **MODELING REAL LIFE** A researcher surveys a random sample of high school students in seven states. The survey asks whether students plan to stay in their home state after graduation. The results, given as joint relative frequencies, are shown in the two-way table. Determine whether planning to stay in their home state and living in Nebraska are independent events.

| | | Location | | |
|---|---|---|---|---|
| | | **Nebraska** | **North Carolina** | **Other States** |
| **Response** | **Yes** | 0.044 | 0.051 | 0.056 |
| | **No** | 0.400 | 0.193 | 0.256 |

**13.** **MP** **PROBLEM SOLVING** You play a game that involves spinning the money wheel shown. You spin the wheel twice. Find the probability that you get more than $500 on your first spin and then go bankrupt on your second spin. ▶ *Example 5*

**14.** **MP** **PROBLEM SOLVING** You play a game that involves drawing two numbers from a hat. There are 25 pieces of paper numbered from 1 to 25 in the hat. Each number is replaced after it is drawn. Find the probability that you will draw the 3 on your first draw and a number greater than 10 on your second draw.

**15.** **MP** **PROBLEM SOLVING** A bag contains 12 movie tickets and 8 concert tickets. You randomly choose 1 ticket and do not replace it. Then you randomly choose another ticket. Find the probability that both events *A* and *B* will occur. ▶ *Example 6*

**Event *A*:** The first ticket is a concert ticket.

**Event *B*:** The second ticket is a concert ticket.

**16.** **MP** **PROBLEM SOLVING** A word game has 100 tiles, 98 of which are letters and 2 of which are blank. The numbers of tiles of each letter are shown. You randomly draw 1 tile, set it aside, and then randomly draw another tile. Find the probability that both events *A* and *B* will occur.

**Event *A*:**
The first tile
is a consonant.

**Event *B*:**
The second tile
is a vowel.

| | | | |
|---|---|---|---|
| A – 9 | H – 2 | O – 8 | V – 2 |
| B – 2 | I – 9 | P – 2 | W – 2 |
| C – 2 | J – 1 | Q – 1 | X – 1 |
| D – 4 | K – 1 | R – 6 | Y – 2 |
| E – 12 | L – 4 | S – 4 | Z – 1 |
| F – 2 | M – 2 | T – 6 | – 2 |
| G – 3 | N – 6 | U – 4 | Blank |

**17.** **ERROR ANALYSIS** A video streaming queue shows 3 animated movies and 4 adventure movies. You randomly choose two movies to watch. Describe and correct the error in finding the probability that both events *A* and *B* occur.

**Event *A*:** The first movie is animated.

**Event *B*:** The second movie is adventure.

✗
$P(A) = \frac{3}{7} \qquad P(B \mid A) = \frac{4}{7}$
$P(A \text{ and } B) = \frac{3}{7} \cdot \frac{4}{7} = \frac{12}{49} \approx 0.245$

**18.** **ERROR ANALYSIS** Events *A* and *B* are independent. Describe and correct the error in finding *P*(*A* and *B*).

✗
$P(A) = 0.6$
$P(B) = 0.2$
$P(A \text{ and } B) = 0.6 + 0.2 = 0.8$

**19.** **MP** **NUMBER SENSE** Events *A* and *B* are independent. Let *P*(*B*) = 0.4 and *P*(*A* and *B*) = 0.13. Find *P*(*A*).

**20.** **MP** **NUMBER SENSE** Events *A* and *B* are dependent. Let *P*(*B*|*A*) = 0.6 and *P*(*A* and *B*) = 0.15. Find *P*(*A*).

**21.** **ANALYZING RELATIONSHIPS** A bin contains orange, blue, green, and yellow water balloons. You randomly select two balloons to toss. Are events *A* and *B* independent or dependent? Explain your reasoning.

**Event *A*:** You choose a green balloon first.

**Event *B*:** You choose a yellow balloon second.

**22.** **HOW DO YOU SEE IT?**
A bag contains one red marble and one blue marble. The diagrams show the possible outcomes of randomly choosing two marbles using different methods. For each method, determine whether the marbles were selected with or without replacement.

a.    1st Draw    2nd Draw      b.    1st Draw    2nd Draw
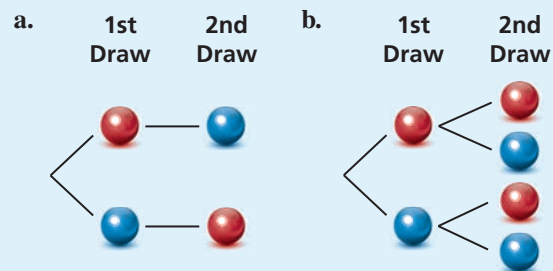
**23.** **MP** **REASONING** You enter to win a contest in which the winners are selected by a random drawing. There is a 5% chance of winning the grand prize, a 20% chance of winning a door prize, and a 1% chance of winning the grand prize and a door prize. Determine whether winning the grand prize and winning a door prize are independent events. Justify your answer.

**24.** **CRITICAL THINKING** You randomly select three cards from a standard deck of 52 playing cards. What is the probability that all three cards are face cards when (a) you replace each card before selecting the next card, and (b) you do not replace each card before selecting the next card?

**25. MAKING AN ARGUMENT** A meteorologist claims that there is a 70% chance of rain. When it rains, there is a 75% chance that your softball game will be rescheduled. Is the game more likely to be rescheduled than played? Explain your reasoning.

**26. THOUGHT PROVOKING**
Two six-sided dice are rolled once. Events $A$ and $B$ are represented by the diagram. Describe each event. Are the two events dependent or independent? Justify your reasoning.
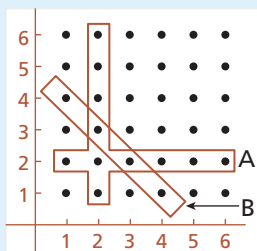


**27. ABSTRACT REASONING** Assume that $A$ and $B$ are independent events.

**a.** Explain why $P(B) = P(B|A)$ and $P(A) = P(A|B)$.

**b.** Can $P(A \text{ and } B)$ also be defined as $P(B) \cdot P(A|B)$? Justify your reasoning.

**28.** **DIG DEEPER** A football team is losing by 14 points near the end of a game. The team scores two touchdowns (worth 6 points each) before the end of the game. After each touchdown, the coach must decide whether to go for 1 point with a kick (which is successful 99% of the time) or 2 points with a run or pass (which is successful 45% of the time).

**a.** If the team goes for 1 point after each touchdown, what is the probability that the team wins? loses? ties?

**b.** If the team goes for 2 points after each touchdown, what is the probability that the team wins? loses? ties?

**c.** Can you develop a strategy so that the team has a probability of winning the game that is greater than the probability of losing? If so, explain your strategy and calculate the probabilities of winning and losing the game.

## REVIEW & REFRESH

**In Exercises 29–32, solve the equation.**

**29.** $2x^2 + 17 = 71$

**30.** $3(4)^{10x} = 48$

**31.** $\log_4 x + \log_4 (x - 12) = 3$

**32.** $0.3x - \frac{3}{5}x + 1.6 = 1.555$

**33.** You roll a six-sided die and flip a coin. Find the probability that you get a 2 when rolling the die and heads when flipping the coin.

**34. MODELING REAL LIFE** A survey asks 68 males and 57 females whether they have been to an escape room. Complete the two-way table. Then interpret the marginal frequencies.

| | | Escape Room | | |
| | | Yes | No | Total |
| --- | --- | --- | --- | --- |
| Gender | Male | 35 | | |
| | Female | | 28 | |
| | Total | | | |

**In Exercises 35 and 36, simplify the expression.**

**35.** $\sqrt[3]{5} \cdot \sqrt[3]{25}$

**36.** $3\sqrt{32} + 6\sqrt{2}$

**37.** Tell whether $x$ and $y$ show *direct variation*, *inverse variation*, or *neither*.

| x | 1 | 2 | 3 | 4 |
| --- | --- | --- | --- | --- |
| y | 12 | 6 | 4 | 3 |

**In Exercises 38 and 39, graph the function. Label the vertex and axis of symmetry.**

**38.** $y = -(x + 6)^2 - 1$

**39.** $f(x) = x^2 + 4x - 5$

**40.** Let $P(A) = 0.4$, $P(B) = 0.88$, and $P(A \text{ and } B) = 0.22$. Find $P(B|A)$ and $P(A|B)$.

**41.** You randomly draw a marble out of a bag containing 8 green marbles, 4 blue marbles, 12 yellow marbles, and 10 red marbles. Find the probability of drawing a marble that is not yellow.

# Probability of Disjoint and Overlapping Events

**Learning Target**   Find probabilities of disjoint and overlapping events.

**Success Criteria**
- I can explain how disjoint events and overlapping events are different.
- I can find probabilities of disjoint events.
- I can find probabilities of overlapping events.
- I can solve real-life problems using more than one probability rule.

## EXPLORE IT! Identifying Overlapping and Disjoint Events

**Work with a partner.**

**Math Practice**

**Understand Mathematical Terms**
What are some other examples of disjoint events and overlapping events that do not involve dice?

**a.** You roll two six-sided dice. Which pair of events below are *overlapping*? Which are *disjoint*? Use Venn diagrams to support your answers.

   **i.** **Event** $A$: The sum is an even number.
   **Event** $B$: The sum is 7.

   **ii.** **Event** $A$: The sum is less than 7.
   **Event** $B$: The sum is a prime number.

**b.** What does it mean for two events to be *overlapping*? *disjoint*?

**c.** Complete the table for each set of events in part (a).

| Experiment | Events in (i) | Events in (ii) |
|---|---|---|
| $P(A)$ | | |
| $P(B)$ | | |
| $P(A$ and $B)$ | | |
| $P(A$ or $B)$ | | |

**d.** Use the results from part (c) to write general formulas for $P(A$ or $B)$.

   **Events are disjoint:**  $P(A$ or $B) =$

   **Events are overlapping:**  $P(A$ or $B) =$

## Compound Events

When you consider only the outcomes shared by both *A* and *B*, you form the *intersection* of *A* and *B*, as shown in the first diagram. Similarly, when you consider all the outcomes that belong to *A*, *B*, or both, you form the *union* of *A* and *B*, as shown in the second diagram. The probability that an outcome is in the union of *A* and *B* is written as *P(A or B)*. The union or intersection of two events is called a **compound event**.



Intersection of *A* and *B*    Union of *A* and *B*    Intersection of *A* and *B* is empty.
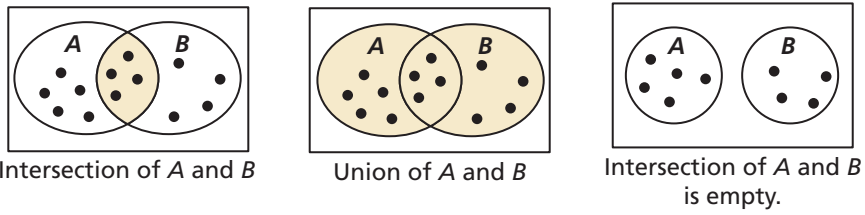
To find *P(A or B)* you must consider the number of outcomes in the intersection of *A* and *B*. Two events are **overlapping** when they have one or more outcomes in common. Two events are **disjoint**, or *mutually exclusive*, when they have no outcomes in common, as shown in the third diagram.

## KEY IDEA

**Probability of Compound Events**

If *A* and *B* are any two events, then the probability of *A* or *B* is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

If *A* and *B* are disjoint events, then $P(A \text{ and } B) = 0$ and the probability of *A* or *B* is

$$P(A \text{ or } B) = P(A) + P(B).$$

**STUDY TIP**

If two events *A* and *B* are overlapping, then the outcomes in the intersection of *A* and *B* are counted *twice* when *P(A)* and *P(B)* are added. So, *P(A and B)* must be subtracted from the sum.

**EXAMPLE 1**    **Finding the Probability of Disjoint Events**    WATCH

A card is randomly selected from a standard deck of 52 playing cards. What is the probability that it is a 10 *or* a face card?

**SOLUTION**

Let event *A* be selecting a 10 and event *B* be selecting a face card. Event *A* has 4 outcomes and event *B* has 12 outcomes. Because *A* and *B* are disjoint, use the disjoint probability formula.
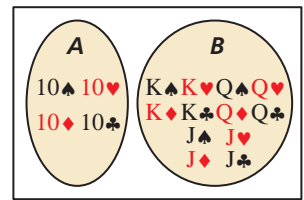


$$P(A \text{ or } B) = P(A) + P(B)$$    Write disjoint probability formula.

$$= \frac{4}{52} + \frac{12}{52}$$    Substitute known probabilities.

$$= \frac{16}{52}$$    Add.

$$= \frac{4}{13} \approx 0.308$$    Simplify and use technology.

▶ So, the probability that a randomly selected card is a 10 or a face card is about 30.8%.

EXAMPLE 2 **Finding the Probability of Overlapping Events** ▶ WATCH

GO DIGITAL

A card is randomly selected from a standard deck of 52 playing cards. What is the probability that it is a face card *or* a spade?

### SOLUTION

**COMMON ERROR**

When two events $A$ and $B$ overlap, as in Example 2, $P(A \text{ or } B)$ does not equal $P(A) + P(B)$.

Let event $A$ be selecting a face card and event $B$ be selecting a spade. Event $A$ has 12 outcomes and event $B$ has 13 outcomes. Of these, 3 outcomes are common to $A$ and $B$. Find $P(A \text{ or } B)$.
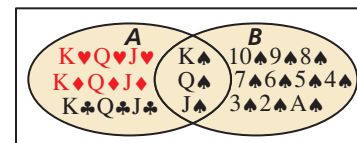
$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$    Write general formula.

$= \dfrac{12}{52} + \dfrac{13}{52} - \dfrac{3}{52}$    Substitute known probabilities.

$= \dfrac{22}{52}$    Add.

$= \dfrac{11}{26} \approx 0.423$    Simplify and use technology.

---

**EXAMPLE 3** **Modeling Real Life** ▶ WATCH

An archaeology lab receives 150 artifacts for processing. Of those, 108 are either made of clay or have coloring. There are 81 artifacts made of clay and 34 artifacts that have coloring. What is the probability that a randomly selected artifact is made of clay *and* has coloring?

### SOLUTION

Let event $A$ be selecting an artifact that is made of clay and event $B$ be selecting an artifact that has coloring. From the given information, you know that $P(A) = \frac{81}{150}$, $P(B) = \frac{34}{150}$, and $P(A \text{ or } B) = \frac{108}{150}$. The probability that a randomly selected artifact is both made of clay *and* has coloring is $P(A \text{ and } B)$.

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$    Write general formula.

$\frac{108}{150} = \frac{81}{150} + \frac{34}{150} - P(A \text{ and } B)$    Substitute known probabilities.

$P(A \text{ and } B) = \frac{81}{150} + \frac{34}{150} - \frac{108}{150}$    Solve for $P(A \text{ and } B)$.

$P(A \text{ and } B) = \frac{7}{150}$    Simplify.

$P(A \text{ and } B) \approx 0.047$    Use technology.

▶ So, the probability that a randomly selected artifact is both made of clay *and* has coloring is about 4.7%.

---

**SELF-ASSESSMENT** | **1** I do not understand. | **2** I can do it with help. | **3** I can do it on my own. | **4** I can teach someone else.

**A card is randomly selected from a standard deck of 52 playing cards. Find the probability of the event.**

**1.** selecting an ace *or* an 8

**2.** selecting a 10 *or* a diamond

**3.** Out of 200 students in a senior class, 113 students are either varsity athletes or on the honor roll. There are 74 seniors who are varsity athletes and 51 seniors who are on the honor roll. What is the probability that a randomly selected senior is both a varsity athlete *and* on the honor roll?

# Using More Than One Probability Rule

The solution to some real-life problems may require the use of two or more probability rules, as shown in the next example.

**EXAMPLE 4**   Modeling Real Life   WATCH   INFO

The American Diabetes Association estimates that 9.4% of people in the United States have diabetes. A medical lab has developed a simple diagnostic test for diabetes that is 98% accurate for people who have the disease and 95% accurate for people who do not have it. The medical lab gives the test to a randomly selected person. What is the probability that the diagnosis is correct?

## SOLUTION

Let event $A$ be "person has diabetes" and event $B$ be "correct diagnosis." Notice that the probability of $B$ depends on the occurrence of $A$, so the events are dependent. When $A$ occurs, $P(B) = 0.98$. When $A$ does not occur, $P(B) = 0.95$.

A probability tree diagram, where the probabilities are given along the branches, can help you see the different ways to obtain a correct diagnosis. Use the complements of events $A$ and $B$ to complete the diagram, where $\overline{A}$ is "person does not have diabetes" and $\overline{B}$ is "incorrect diagnosis." Notice that the probabilities for all branches from the same point must sum to 1.
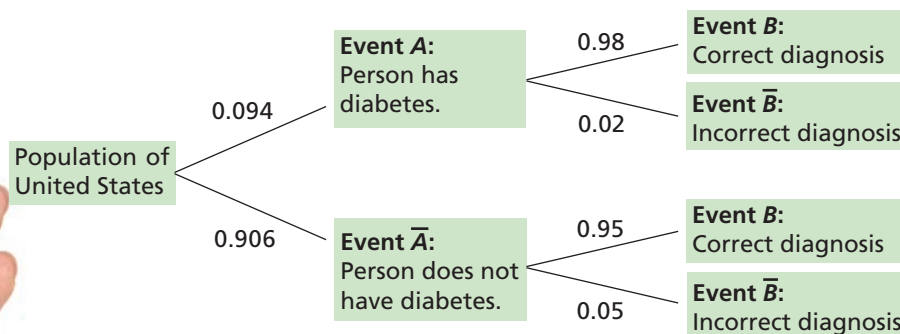


To find the probability that the diagnosis is correct, follow the branches leading to event $B$.

$$P(B) = P(A \text{ and } B) + P(\overline{A} \text{ and } B)$$   Use tree diagram.

$$= P(A) \cdot P(B|A) + P(\overline{A}) \cdot P(B|\overline{A})$$   Probability of dependent events

$$= (0.094)(0.98) + (0.906)(0.95)$$   Substitute.

$$\approx 0.953$$   Use technology.

▶  The probability that the diagnosis is correct is about 0.953, or 95.3%.

---

**SELF-ASSESSMENT**   1 I do not understand.   2 I can do it with help.   3 I can do it on my own.   4 I can teach someone else.

4. In Example 4, what is the probability that a randomly selected person diagnosed with diabetes actually has the disease?

5. A high school basketball team leads at halftime in 60% of the games in a season. The team wins 80% of the time when they have the halftime lead, but only 10% of the time when they do not. What is the probability that the team wins a particular game during the season?

In Exercises 1 and 2, events *A* and *B* are disjoint. Find *P(A or B)*.

**1.** $P(A) = 0.3$, $P(B) = 0.1$  **2.** $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{5}$

In Exercises 3–8, each section of the spinner shown has the same area. Find the probability of the event.
▶ *Examples 1 and 2*



**3.** spinning blue *or* a 1

**4.** spinning green *or* a multiple of 4

**5.** spinning red *or* an odd number

**6.** spinning yellow *or* a number less than 5

**7.** spinning a factor of 6 *or* a number greater than 9

**8.** spinning an even number *or* a prime number

**ERROR ANALYSIS** In Exercises 9 and 10, describe and correct the error in finding the probability of randomly drawing the given card from a standard deck of 52 playing cards.

**9.**

✗
P(heart or face card)
= P(heart) + P(face card)
$= \frac{13}{52} + \frac{12}{52} = \frac{25}{52}$

**10.**

✗
P(club or 9)
= P(club) + P(9) + P(club and 9)
$= \frac{13}{52} + \frac{4}{52} + \frac{1}{52} = \frac{9}{26}$

**11.** **MP** **PROBLEM SOLVING** You perform an experiment for your science project to determine how well plants grow under different light sources. Of the 30 plants in the experiment, 12 receive visible light, 15 receive ultraviolet light, and 6 receive both visible and ultraviolet light. What is the probability that a plant in the experiment receives visible *or* ultraviolet light?

**12.** **MP** **PROBLEM SOLVING** Of 162 students honored at an academic awards banquet, 48 won awards for mathematics and 78 won awards for English. There are 14 students who won awards for both mathematics and English. A student is selected at random for an interview. What is the probability that the student won an award for English *or* mathematics?

**13.** **MODELING REAL LIFE** A group of 40 trees in a forest are not growing properly. A botanist determines that 34 of the trees have a disease or are being damaged by insects, with 18 trees having a disease and 20 being damaged by insects. What is the probability that a randomly selected tree has both a disease *and* is being damaged by insects? ▶ *Example 3*

**14.** **MODELING REAL LIFE** Out of 55 teenagers enrolled in dance classes, 30 teenagers take either hip hop or jazz classes. There are 13 teenagers who take hip hop classes and 24 teenagers who take jazz classes. What is the probability that a randomly selected teenager takes both hip hop *and* jazz classes?

**15.** **DRAWING CONCLUSIONS** A company is focus testing a new type of fruit drink. The focus group is 47% male. Of the responses, 40% of the males and 54% of the females said they would buy the fruit drink. What is the probability that a randomly selected person would buy the fruit drink? ▶ *Example 4*

**16.** **DRAWING CONCLUSIONS** The Redbirds trail the Bluebirds by one goal with 1 minute left in the hockey game. The Redbirds' coach must decide whether to remove the goalie and add a frontline player. The probabilities of each team scoring are shown in the table.

|  | Goalie | No goalie |
|---|---|---|
| **Redbirds score** | 0.1 | 0.3 |
| **Bluebirds score** | 0.1 | 0.6 |

**a.** Find the probability that the Redbirds score and the Bluebirds do not score when the coach leaves the goalie in.

**b.** Find the probability that the Redbirds score and the Bluebirds do not score when the coach takes the goalie out.

**c.** Based on parts (a) and (b), what should the coach do?

**17.** **MP** **PROBLEM SOLVING** You can win concert tickets from a radio station if you are the first person to call when the song of the day is played, or if you are the first person to correctly answer a daily trivia question. The song of the day is announced at a random time between 7:00 and 7:30 A.M. The trivia question is asked at a random time between 7:15 and 7:45 A.M. You begin listening to the radio station at 7:20 A.M. Find the probability that you miss the announcement of the song of the day *or* the trivia question.

**18.** **HOW DO YOU SEE IT?**
Are events *A* and *B* disjoint events? Explain your reasoning.
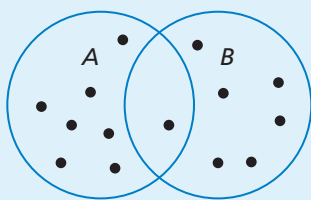
**19.** **MP** **PROBLEM SOLVING** You take a bus from your neighborhood to a store. The express bus arrives at your neighborhood at a random time between 7:30 and 7:36 A.M. The local bus arrives at your neighborhood at a random time between 7:30 and 7:40 A.M. You arrive at the bus stop at 7:33 A.M. Find the probability that you miss the express bus *or* the local bus.

**20.** **THOUGHT PROVOKING**
Write a general rule for finding $P(A$ or $B$ or $C)$ for (a) disjoint and (b) overlapping events *A, B,* and *C*.

## REVIEW & REFRESH

**In Exercises 21 and 22, write the function in vertex form. Then identify the vertex.**

**21.** $y = x^2 + 6x - 4$

**22.** $y = 3x^2 - 12x + 11$

**23.** You randomly draw a card from a standard deck of 52 playing cards, set it aside, and then randomly draw another card from the deck. Find the probability that both cards are hearts.

**24.** Use the graph to solve $\log_5 (2x - 1) = 1$.
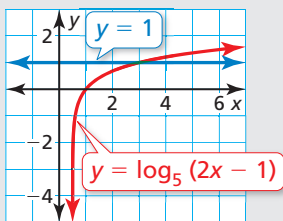
**25.** **MODELING REAL LIFE** A group of people wants to equally share the cost of a gift. The cost per person *c* (in dollars) varies inversely with the number *n* of people sharing the cost. When 3 people share the cost, the cost per person is $40. What is the cost per person when 5 people share the cost?

**In Exercises 26 and 27, find the missing probability.**

**26.** $P(A) = 0.6$
$P(A$ and $B) = 0.12$
$P(B|A) = $ _____

**27.** $P(A) = 0.53$
$P(B) = 0.27$
$P(A$ and $B) = 0.16$
$P(A$ or $B) = $ _____

**In Exercises 28 and 29, find the inverse of the function.**

**28.** $f(x) = \frac{1}{4}x + 3$

**29.** $f(x) = \sqrt{x + 7}$

**30.** A teacher asks students to vote on whether they want to take a test Friday or Monday. The results are shown in the two-way table. Make a two-way table that shows the joint and marginal relative frequencies. Then interpret one of the joint relative frequencies and one of the marginal relative frequencies.

| | | Vote | |
| --- | --- | --- | --- |
| | | **Friday** | **Monday** |
| **Gender** | **Male** | 7 | 8 |
| | **Female** | 5 | 6 |

# 8.6 Permutations and Combinations

**Learning Target**  Count permutations and combinations.

**Success Criteria**
- I can explain the difference between permutations and combinations.
- I can find numbers of permutations and combinations.
- I can find probabilities using permutations and combinations.

## EXPLORE IT! Counting Outcomes

**Work with a partner.**

**Math Practice**

**Reason Abstractly**
In how many different orders can $n$ dogs finish in a race? Explain your reasoning.

**a.** A fair conducts three obstacle course races. In how many different orders can the dogs finish in each race? Justify your answers.

**Race 1**

Labrador Retriever    Golden Retriever

**Race 2**

Golden Retriever    Labrador Retriever    German Shepherd

**Race 3**

Dalmation    German Shepherd    Golden Retriever    Labrador Retriever

**b.** For each race in part (a), in how many different ways can the dogs finish first and second? Justify your answers.

**c.** For each race in part (a), how many different pairs of dogs can you form?

**d.** Explain why your answers in part (c) are different from your answers in part (b).
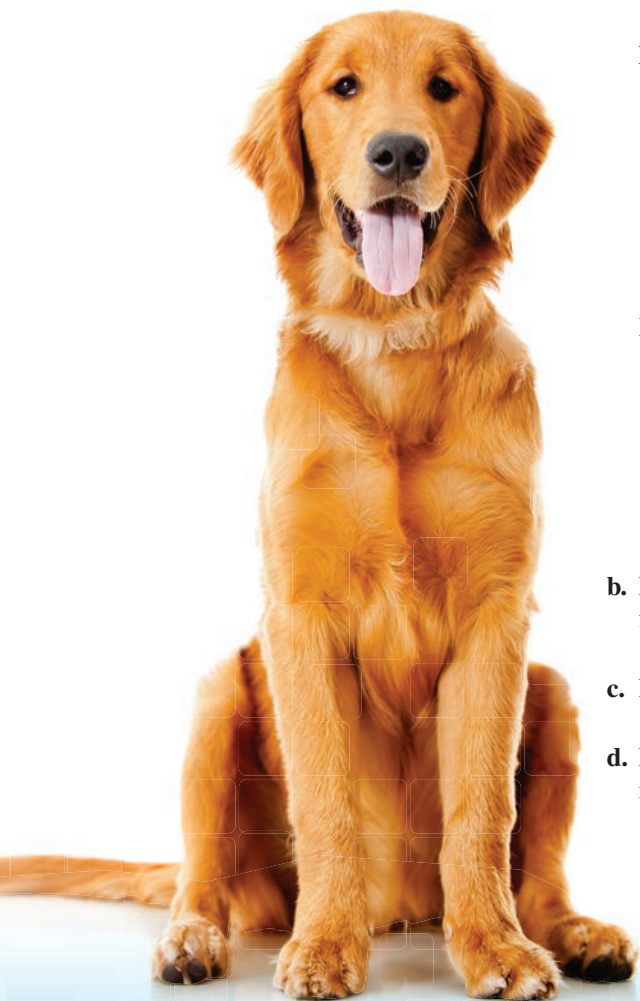
## Permutations

A **permutation** is an arrangement of objects in which order is important. For instance, the 6 possible permutations of the letters A, B, and C are shown.

ABC   ACB   BAC   BCA   CAB   CBA

**Vocabulary**

permutation, *p. 440*
*n* factorial, *p. 440*
combination, *p. 442*
Binomial Theorem, *p. 444*

### EXAMPLE 1    Counting Permutations

WATCH

Consider the letters in the word JULY.

**a.** In how many ways can you arrange all of the letters?

**b.** In how many ways can you arrange 2 of the letters?

### SOLUTION

**REMEMBER**

*Fundamental Counting Principle*: If one event can occur in *m* ways and another event can occur in *n* ways, then the number of ways that both events can occur is *m* • *n*. The Fundamental Counting Principle can be extended to three or more events.

**a.** Because the order of the letters is important, use the Fundamental Counting Principle to find the number of permutations of the letters in the word JULY.

$$\text{Number of permutations} = \left(\begin{matrix}\text{Choices for}\\\text{1st letter}\end{matrix}\right)\left(\begin{matrix}\text{Choices for}\\\text{2nd letter}\end{matrix}\right)\left(\begin{matrix}\text{Choices for}\\\text{3rd letter}\end{matrix}\right)\left(\begin{matrix}\text{Choices for}\\\text{4th letter}\end{matrix}\right)$$

$$= 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 24$$

▶ There are 24 ways you can arrange all of the letters in the word JULY.

**b.** When arranging 2 letters of the word JULY, you have 4 choices for the first letter and 3 choices for the second letter.

$$\text{Number of permutations} = \left(\begin{matrix}\text{Choices for}\\\text{1st letter}\end{matrix}\right)\left(\begin{matrix}\text{Choices for}\\\text{2nd letter}\end{matrix}\right)$$

$$= 4 \cdot 3$$

$$= 12$$

▶ There are 12 ways you can arrange 2 of the letters in the word JULY.

In Example 1(a), you evaluated the expression 4 • 3 • 2 • 1. This expression can be written as 4! and is read "4 *factorial*." For any positive integer *n*, the product of the integers from 1 to *n* is called **n factorial** and is written as

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1.$$

As a special case, the value of 0! is defined to be 1.

In Example 1(b), you found the permutations of 4 objects taken 2 at a time. You can also find the number of permutations using the following formula.

### KEY IDEA

**Permutations**

**Formula**

The number of permutations of *n* objects taken *r* at a time, where $r \le n$, is given by

$$_nP_r = \frac{n!}{(n - r)!}.$$

**Example**

The number of permutations of 4 objects taken 2 at a time is

$$_4P_2 = \frac{4!}{(4 - 2)!} = \frac{4 \cdot 3 \cdot 2!}{2!} = 12.$$

**EXAMPLE 2**   **Using the Permutations Formula**   **WATCH**

Ten horses run in a race. In how many different ways can the horses finish first, second, and third? (Assume there are no ties.)

**SOLUTION**

To find the number of permutations of 3 horses chosen from 10, find $_{10}P_3$.

$$_{10}P_3 = \frac{10!}{(10-3)!}$$   Permutations formula

$$= \frac{10!}{7!}$$   Subtract.

$$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}}$$   Expand 10!. Divide out the common factor, 7!.

$$= 720$$   Simplify.

▶   There are 720 ways for the horses to finish first, second, and third.

**Check**

nPr(10,3)

= 720

**EXAMPLE 3**   **Finding a Probability Using Permutations**   **WATCH**

You ride on a float with your soccer team in a parade. There are 12 floats in the parade, and their order is chosen at random. Find the probability that your float is first and the float with the school chorus is second.

**SOLUTION**

**Step 1**  Write the number of possible ways that two of the floats can be first and second as the number of permutations of the 12 floats taken 2 at a time.

$$_{12}P_2 = \frac{12!}{(12-2)!}$$   Permutations formula

$$= \frac{12!}{10!}$$   Subtract.

$$= \frac{12 \cdot 11 \cdot \cancel{10!}}{\cancel{10!}}$$   Expand 12!. Divide out the common factor, 10!.

$$= 132$$   Simplify.

**Step 2**  Find the number of favorable outcomes.

Only one of the possible permutations includes your float first and the float with the school chorus second.

**Step 3**  Find the probability.

$$P(\text{soccer team is 1st, chorus is 2nd}) = \frac{1}{132}$$

▶   The probability is $\frac{1}{132}$.

**SELF-ASSESSMENT**   **1** I do not understand.   **2** I can do it with help.   **3** I can do it on my own.   **4** I can teach someone else.

1. Consider the letters in the word MARCH. In how many ways can you arrange (a) all of the letters and (b) 3 of the letters?

2. **WHAT IF?** In Example 3, there are 14 floats in the parade. Find the probability that the soccer team is first and the chorus is second.

3. **MP STRUCTURE** Find the number of permutations of $n$ objects taken $n$ at a time. Justify your answer.

EXAMPLE 6 **Finding a Probability Using Combinations**  ▶ WATCH  GO DIGITAL

A yearbook editor has selected 14 photos, including one of you and one of your friend, to use in a collage for the yearbook. The photos are placed at random. There is room for 2 photos at the top of the page. What is the probability that your photo and your friend's photo are the 2 placed at the top of the page?

## SOLUTION

1. **Understand the Problem** You are given the total number of photos in the collage and the number of photos placed at the top of the page. You are asked to find the probability that your photo and your friend's photo are placed at the top of the page.

2. **Make a Plan** The order in which the photos are chosen is not important. Find the number of possible outcomes and the number of favorable outcomes. Then use these numbers to find the probability.

3. **Solve and Check**

   **Step 1** Write the number of possible outcomes as the number of combinations of 14 photos taken 2 at a time, or $_{14}C_2$.

$$_{14}C_2 = \frac{14!}{(14 - 2)! \cdot 2!} \qquad \text{Combinations formula}$$

$$= \frac{14!}{12! \cdot 2!} \qquad \text{Subtract.}$$

$$= \frac{14 \cdot 13 \cdot \cancel{12!}}{\cancel{12!} \cdot (2 \cdot 1)} \qquad \text{Expand 14! and 2!. Divide out the common factor, 12!.}$$

$$= 91 \qquad \text{Simplify.}$$

**Check**

| nCr(14,2) | = 91 |
|---|---|

   **Step 2** Find the number of favorable outcomes.

   Only one of the possible combinations includes your photo and your friend's photo.

   **Step 3** Find the probability.

$$P(\text{your photo and your friend's photo are chosen}) = \frac{1}{91}$$

▶ The probability is $\frac{1}{91}$.

---

**SELF-ASSESSMENT** | **1** I do not understand. | **2** I can do it with help. | **3** I can do it on my own. | **4** I can teach someone else.

4. Count the possible combinations of 3 letters chosen from the list A, B, C, D, E.

5. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$$\frac{7!}{2! \cdot 5!} \qquad\qquad _7C_5 \qquad\qquad _7C_2 \qquad\qquad \frac{7!}{(7 - 2)!}$$

6. **WHAT IF?** In Example 5, you can choose 3 side dishes out of the list of 8 side dishes. How many combinations are possible?

7. **WHAT IF?** In Example 6, there are 20 photos in the collage. Find the probability that your photo and your friend's photo are the 2 placed at the top of the page.

EXAMPLE 6 **Finding a Probability Using Combinations** ▶ WATCH

GO DIGITAL

A yearbook editor has selected 14 photos, including one of you and one of your friend, to use in a collage for the yearbook. The photos are placed at random. There is room for 2 photos at the top of the page. What is the probability that your photo and your friend's photo are the 2 placed at the top of the page?

### SOLUTION

1. **Understand the Problem** You are given the total number of photos in the collage and the number of photos placed at the top of the page. You are asked to find the probability that your photo and your friend's photo are placed at the top of the page.

2. **Make a Plan** The order in which the photos are chosen is not important. Find the number of possible outcomes and the number of favorable outcomes. Then use these numbers to find the probability.

3. **Solve and Check**

    **Step 1** Write the number of possible outcomes as the number of combinations of 14 photos taken 2 at a time, or $_{14}C_2$.

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$$= \frac{14!}{12! \cdot 2!} \qquad \text{Subtract.}$$

$$= \frac{14 \cdot 13 \cdot \cancel{12!}}{\cancel{12!} \cdot (2 \cdot 1)} \qquad \text{Expand 14! and 2!. Divide out the common factor, 12!.}$$

$$= 91 \qquad \text{Simplify.}$$

**Check**

| nCr(14,2) | = 91 |
|---|---|

    **Step 2** Find the number of favorable outcomes.

    Only one of the possible combinations includes your photo and your friend's photo.

    **Step 3** Find the probability.

$$P(\text{your photo and your friend's photos are chosen}) = \frac{1}{91}$$

▶ The probability is $\frac{1}{91}$.

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**SELF-ASSESSMENT** **1** I do not understand. **2** I can do it with help. **3** I can do it on my own. **4** I can teach someone else.

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$$\frac{7!}{2! \cdot 5!} \qquad _7C_5 \qquad _7C_2 \qquad \frac{7!}{(7-2)!}$$

6. **WHAT IF?** In Example 5, you can choose 3 side dishes out of the list of 8 side dishes. How many combinations are possible?

7. **WHAT IF?** In Example 6, there are 20 photos in the collage. Find the probability that your photo and your friend's photo are the 2 placed at the top of the page.

## Binomial Expansions

In Section 4.2, you used Pascal's Triangle to find binomial expansions. The table shows that the coefficients in the expansion of $(a + b)^n$ correspond to combinations.

| | $n$ | Pascal's Triangle as Numbers | Pascal's Triangle as Combinations | Binomial Expansion |
|---|---|---|---|---|
| 0th row | 0 | 1 | $_0C_0$ | $(a + b)^0 = \quad 1$ |
| 1st row | 1 | 1  1 | $_1C_0 \quad _1C_1$ | $(a + b)^1 = \quad 1a + 1b$ |
| 2nd row | 2 | 1  2  1 | $_2C_0 \quad _2C_1 \quad _2C_2$ | $(a + b)^2 = \quad 1a^2 + 2ab + 1b^2$ |
| 3rd row | 3 | 1  3  3  1 | $_3C_0 \quad _3C_1 \quad _3C_2 \quad _3C_3$ | $(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$ |

The results in the table are generalized in the **Binomial Theorem**.

### KEY IDEA

**The Binomial Theorem**

For any positive integer $n$, the binomial expansion of $(a + b)^n$ is

$$(a + b)^n = {}_nC_0\, a^n b^0 + {}_nC_1\, a^{n-1}b^1 + {}_nC_2\, a^{n-2}b^2 + \cdots + {}_nC_n\, a^0 b^n.$$

Notice that each term in the expansion of $(a + b)^n$ has the form ${}_nC_r\, a^{n-r}b^r$, where $r$ is an integer from 0 to $n$.

### EXAMPLE 7   Using the Binomial Theorem

WATCH

**a.** Use the Binomial Theorem to write the expansion of $(x^2 + y)^3$.

**b.** Find the coefficient of $x^4$ in the expansion of $(3x + 2)^{10}$.

### SOLUTION

**a.** $(x^2 + y)^3 = {}_3C_0(x^2)^3 y^0 + {}_3C_1(x^2)^2 y^1 + {}_3C_2(x^2)^1 y^2 + {}_3C_3(x^2)^0 y^3$

$= (1)(x^6)(1) + (3)(x^4)(y^1) + (3)(x^2)(y^2) + (1)(1)(y^3)$

$= x^6 + 3x^4y + 3x^2y^2 + y^3$

**b.** From the Binomial Theorem, you know

$$(3x + 2)^{10} = {}_{10}C_0(3x)^{10}(2)^0 + {}_{10}C_1(3x)^9(2)^1 + \cdots + {}_{10}C_{10}(3x)^0(2)^{10}.$$

Each term in the expansion has the form ${}_{10}C_r(3x)^{10-r}(2)^r$. The term containing $x^4$ occurs when $r = 6$.

$${}_{10}C_6(3x)^4(2)^6 = (210)(81x^4)(64) = 1{,}088{,}640x^4$$

▶ The coefficient of $x^4$ is 1,088,640.

---

**SELF-ASSESSMENT**  **1** I do not understand.  **2** I can do it with help.  **3** I can do it on my own.  **4** I can teach someone else.

**8.** Use the Binomial Theorem to write the expansion of (a) $(x + 3)^5$ and (b) $(2p - q)^4$.

**9.** Find the coefficient of $x^5$ in the expansion of $(x - 3)^7$.

**10.** Find the coefficient of $x^3$ in the expansion of $(2x + 5)^8$.

**In Exercises 1–6, find the number of ways you can arrange (a) all of the letters and (b) 2 of the letters in the given word.** ▶ *Example 1*

1. AT
2. TRY
3. ROCK
4. WATER
5. FAMILY
6. FLOWERS

**In Exercises 7–12, evaluate the expression.**

7. $_5P_2$
8. $_7P_3$
9. $_9P_1$
10. $_6P_5$
11. $_8P_6$
12. $_{12}P_0$

13. **MODELING REAL LIFE** Eleven students are competing in a graphic design contest. In how many different ways can the students finish first, second, and third? ▶ *Example 2*

14. **MODELING REAL LIFE** Six friends go to a movie theater. In how many different ways can they sit together in a row of 6 empty seats?

15. **MODELING REAL LIFE** You and your friend are 2 of 8 servers working a shift in a restaurant. At the beginning of the shift, the manager randomly assigns one section to each server. Find the probability that you are assigned Section 1 and your friend is assigned Section 2. ▶ *Example 3*

16. **MODELING REAL LIFE** You make 6 posters to hold up at a basketball game. Each poster has a letter of the word TIGERS. You and 5 friends sit next to each other in a row. The posters are distributed at random. Find the probability that TIGERS is spelled correctly when you hold up the posters.

**In Exercises 17–20, count the possible combinations of $r$ letters chosen from the given list.** ▶ *Example 4*

17. A, B, C, D; $r = 3$
18. L, M, N, O; $r = 2$
19. U, V, W, X, Y, Z; $r = 3$
20. D, E, F, G, H; $r = 4$

**In Exercises 21–26, evaluate the expression.**

21. $_5C_1$
22. $_8C_5$
23. $_9C_9$
24. $_8C_6$
25. $_{12}C_3$
26. $_{11}C_4$

27. **MODELING REAL LIFE** A team of 25 rowers attends a rowing tournament. Five rowers compete at a time. How many combinations of 5 rowers are possible? ▶ *Example 5*

28. **MODELING REAL LIFE** A grocery store sells 7 different flavors of vegetable dip. You have enough money to purchase 2 flavors. How many combinations of 2 flavors of vegetable dip are possible?

**ERROR ANALYSIS** In Exercises 29 and 30, describe and correct the error in evaluating the expression.

29.

✗ $_{11}P_7 = \dfrac{11!}{(11-7)} = \dfrac{11!}{4} = 9,979,200$

30.

✗ $_9C_4 = \dfrac{9!}{(9-4)!} = \dfrac{9!}{5!} = 3024$

31. **COLLEGE PREP** Which expression has the greatest value, where $n > 4$?

Ⓐ $_nP_{n-1}$  Ⓑ $_nP_{n-2}$

Ⓒ $_nP_{n-3}$  Ⓓ $_nP_{n-4}$

32. **COLLEGE PREP** What is the solution of $_nC_{n-2} = {}_nC_{n-3}$?

Ⓐ $n = 3$  Ⓑ $n = 5$

Ⓒ $n = 6$  Ⓓ $n = 14$

**MP** **REASONING** In Exercises 33–36, tell whether the question can be answered using *permutations* or *combinations*. **Explain your reasoning. Then answer the question.**

**33.** To complete an exam, you must answer 8 questions from a list of 10 questions. In how many ways can you complete the exam?

**34.** Ten students are auditioning for 3 different roles in a play. In how many ways can the 3 roles be filled?

**35.** Fifty-two athletes are competing in a bicycle race. In how many orders can the bicyclists finish first, second, and third? (Assume there are no ties.)

**36.** An employee at a pet store needs to catch 5 tetras in an aquarium containing 27 tetras. In how many groupings can the employee capture 5 tetras?

**37.** **MODELING REAL LIFE** You and your friend are in the studio audience on a game show. From an audience of 300 people, 2 people are randomly selected as contestants. What is the probability that you and your friend are chosen? ▶ *Example 6*

**38.** **MODELING REAL LIFE** You work 5 evenings each week at a bookstore. Your supervisor assigns you 5 evenings at random from the 7 possibilities. What is the probability that your schedule does not include working on the weekend?

**39.** **MP** **PATTERNS** Describe why it makes sense to define 0! as 1.

**40.** **CRITICAL THINKING** Show that each identity is true for any whole numbers $r$ and $n$, where $0 \leq r \leq n$.

   **a.** $_nC_n = 1$

   **b.** $_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

**41.** **OPEN-ENDED** Describe a real-life situation where the number of possibilities is given by $_5P_2$. Then describe a real-life situation where the number of possibilities is given by $_5C_2$.

**42.** **MP** **REASONING** Consider a set of 4 objects.

   **a.** Are there more permutations of all 4 of the objects or of 3 of the objects? Explain your reasoning.

   **b.** Are there more combinations of all 4 of the objects or of 3 of the objects? Explain your reasoning.

   **c.** Compare your answers to parts (a) and (b).

**43.** **MP** **REPEATED REASONING** Complete the table for each given value of $r$. Then write an inequality relating $_nP_r$ and $_nC_r$. Explain your reasoning.

| | $r = 0$ | $r = 1$ | $r = 2$ | $r = 3$ |
|---|---|---|---|---|
| $_3P_r$ | | | | |
| $_3C_r$ | | | | |

**44.** **MP** **REASONING** Write an equation that relates $_nP_r$ and $_nC_r$. Then use your equation to find and interpret the value of $\dfrac{_{182}P_4}{_{182}C_4}$.

**MP** **REASONING** In Exercises 45 and 46, find the probability of winning a lottery using the given rules. Assume that lottery numbers are selected at random.

**45.** You must correctly select 6 numbers, each an integer from 0 to 49. The order is not important.

**46.** You must correctly select 4 numbers, each an integer from 0 to 9. The order is important.

In Exercises 47–54, use the Binomial Theorem to write the binomial expansion. ▶ *Example 7a*

**47.** $(x + 2)^3$

**48.** $(c - 4)^5$

**49.** $(a + 3b)^4$

**50.** $(4p - q)^6$

**51.** $(w^3 - 3)^4$

**52.** $(2s^4 + 5)^5$

**53.** $(3u + v^2)^6$

**54.** $(x^3 - y^2)^4$

In Exercises 55–60, use the given value of $n$ to find the coefficient of $x^n$ in the expansion of the binomial. ▶ *Example 7b*

**55.** $(x - 2)^{10}, n = 5$

**56.** $(x - 3)^7, n = 4$

**57.** $(2x + 5)^{12}, n = 7$

**58.** $(3x - 1)^9, n = 2$

**59.** $\left(\frac{1}{2}x - 4\right)^{11}, n = 4$
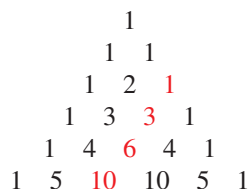
**60.** $\left(\frac{1}{4}x + 6\right)^6, n = 3$

**61.** **MP** **REASONING** Write the eighth row of Pascal's Triangle as combinations and as numbers.

**62.** **MP** **PROBLEM SOLVING** The triangular numbers are a sequence of numbers found in Pascal's Triangle. The first four triangular numbers are 1, 3, 6, and 10.

  **a.** Use Pascal's Triangle to write the first four triangular numbers as combinations.

$$
\begin{array}{ccccccccccc}
&&&&& 1 \\
&&&& 1 && 1 \\
&&& 1 && 2 && 1 \\
&& 1 && 3 && 3 && 1 \\
& 1 && 4 && 6 && 4 && 1 \\
1 && 5 && 10 && 10 && 5 && 1
\end{array}
$$

  **b.** Use your result from part (a) to write an explicit rule for the $n$th triangular number $T_n$.

**63.** **CONNECTING CONCEPTS**
A polygon is convex when no line that contains a side of the polygon contains a point in the interior of the polygon. Consider a convex polygon with $n$ sides.
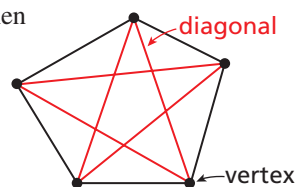

diagonal
vertex

  **a.** Use the combinations formula to write an expression for the number of diagonals in an $n$-sided polygon.

  **b.** Use your result from part (a) to write a formula for the number of diagonals of an $n$-sided convex polygon.

**64.** **MP** **PROBLEM SOLVING** You are ordering a burrito with 2 main ingredients and 3 toppings. The menu below shows the possible choices. How many different burritos are possible?



MAIN INGREDIENTS
CHICKEN
STEAK
PORK
BLACK BEANS
RED BEANS
RICE

TOPPINGS
* PEPPERS
* GUACAMOLE
* JALAPEÑOS
* SALSA
* EXTRA CHEESE
* BLACK OLIVES
* ONIONS
* PICO DE GALLO

**65.** **MP** **PROBLEM SOLVING** You select 2 different contemporary music playlists and 1 classical music playlist from the music collection shown. How many different sets of playlists can you select?



CONTEMPORARY
Country    Jazz
Rap        Rock & Roll
CLASSICAL
Opera      Symphony
Concerto
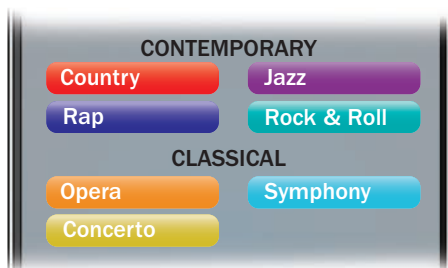
**66.** **MP** **PROBLEM SOLVING** Every student in your history class is required to present a project in front of the class. Each day, 4 students present in an order chosen at random by the teacher. You present on the first day.

  **a.** What is the probability that you are chosen to be the first or second presenter on the first day?

  **b.** What is the probability that you are chosen to be the second or third presenter on the first day? Compare your answer with that in part (a).

**67.** **MP** **PROBLEM SOLVING** The organizer of a cast party for a drama club asks each of the 6 cast members to bring 1 food item from a list of 10 items. Assuming each member randomly chooses a food item to bring, what is the probability that at least 2 of the 6 cast members bring the same item?

**68.** **HOW DO YOU SEE IT?**
A bag contains one green marble, one red marble, and one blue marble. The diagram shows the possible outcomes of randomly drawing three marbles from the bag without replacement.

**1st Draw   2nd Draw   3rd Draw**
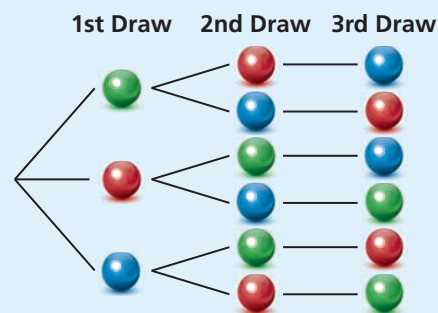


  **a.** How many combinations of three marbles can be drawn from the bag? Explain.

  **b.** How many permutations of three marbles can be drawn from the bag? Explain.

**69.** **MP** **PROBLEM SOLVING** You and a friend are two of 10 students performing in a school talent show. The order of the performances is determined at random. The first 5 performers go on stage before the intermission.

  **a.** What is the probability that you are the last performer before the intermission and your friend performs immediately before you?

  **b.** What is the probability that you are *not* the first performer?

**70.** **THOUGHT PROVOKING**
How many integers, greater than 999 but not greater than 4000, can be formed with the digits 0, 1, 2, 3, and 4? Repetition of digits is allowed.

**71.** **MP** **PROBLEM SOLVING** Consider a standard deck of 52 playing cards. The order in which the cards are dealt for a "hand" does not matter. How many different 5-card hands have all 5 cards of a single suit?

**72.** **MP** **PROBLEM SOLVING** There are 30 students in your class. Your science teacher chooses 5 students at random to complete a group project. Find the probability that you and your 2 best friends in the science class are chosen to work in the group. Explain how you found your answer.

**73.** **CONNECTING CONCEPTS** Use combinations to determine how many lines can be drawn through each pair of points below.

$$E\bullet$$
$$A\bullet \quad B\bullet$$
$$\quad C\bullet \quad D\bullet$$
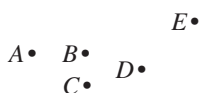
**74.** **DIG DEEPER** Follow the steps below to explore a famous probability problem called the *birthday problem*. (Assume there are 365 equally likely birthdays possible.)

**a.** What is the probability that at least 2 people share the same birthday in a group of 6 randomly chosen people? in a group of 10 randomly chosen people?

**b.** Generalize the results from part (a) by writing a formula for the probability $P(n)$ that at least 2 people in a group of $n$ people share the same birthday. (*Hint:* Use $_nP_r$ notation in your formula.)

**c.** Use technology to determine by what group size the probability that at least 2 people share the same birthday first exceeds 50%.

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## REVIEW & REFRESH

**75.** Find the dimensions of the cube.
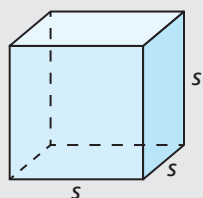


Volume = 343 cm³

**76.** Graph $f(x) = |x - 2| + 1$. Then describe the transformations from the graph of $f(x) = |x|$ to the graph of the function.

**In Exercises 77 and 78, use the Binomial Theorem to write the binomial expansion**

**77.** $(x + 6)^3$         **78.** $(2x - 3y)^3$

**79.** You spin the wheel shown. Each section of the wheel has the same area. Use a sample space to determine whether randomly spinning red and then blue are independent events.
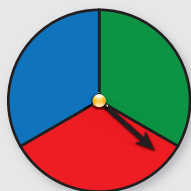


**80.** **MP** **NUMBER SENSE** Events $A$ and $B$ are dependent. Suppose $P(A \text{ and } B) = 0.04$ and $P(A) = 0.16$. Find $P(B|A)$.
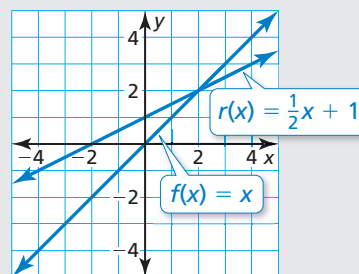
**In Exercises 81 and 82, graph the function. Find the domain and range of the function.**

**81.** $g(x) = -(x)^{1/4} + 2$     **82.** $f(x) = \frac{1}{2}\sqrt[3]{x + 1}$

**83.** Determine the type of function represented by the table. Explain your reasoning.

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $y$ | 1 | 2 | 4 | 8 | 16 | 32 |

**84.** Use the graphs of $f$ and $r$ to describe the transformation from the graph of $f$ to the graph of $r$.



$r(x) = \frac{1}{2}x + 1$

$f(x) = x$

**85.** Solve the equation $x^2 + 3x = 5x^2 + 2x$ by graphing.

**86.** Events $A$ and $B$ are disjoint. Find $P(A \text{ or } B)$ when $P(A) = 0.4$ and $P(B) = 0.6$.

**87.** Determine whether the function $y = (1.5)^x$ represents *exponential growth* or *exponential decay*. Then graph the function.

# 8.7 Binomial Distributions

**Learning Target**    Understand binomial distributions.

**Success Criteria**
- I can explain the meaning of a probability distribution.
- I can construct and interpret probability distributions.
- I can find probabilities using binomial distributions.
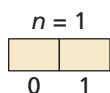
## EXPLORE IT! Counting Outcomes

**Work with a partner.** The diagrams represent the possible outcomes when flipping $n$ coins.
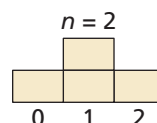
**STUDY TIP**

When 4 coins are flipped, the possible outcomes are

| TTTT | TTTH | TTHT | THTT |
| HTTT | TTHH | THTH | THHT |
| HTTH | HTHT | HHTT | THHH |
| HTHH | HHTH | HHHT | HHHH. |

The diagram shows the numbers of outcomes having 0, 1, 2, 3, and 4 heads.

$n = 1$

0    1
**Number of Heads**

$n = 2$

0    1    2
**Number of Heads**

$n = 3$

0    1    2    3
**Number of Heads**

$n = 4$
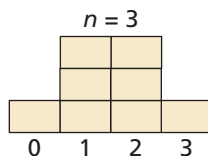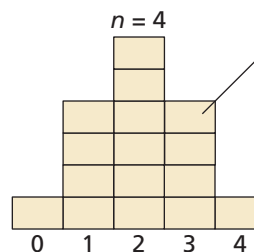
0    1    2    3    4
**Number of Heads**

Flipping 3 heads can occur 4 different ways. For example:

a. What is the probability of flipping 2 heads when 4 coins are flipped? Explain your reasoning.

b. Draw a similar diagram that represents the possible outcomes when flipping 5 coins. What is the probability of flipping 2 heads when 5 coins are flipped?

**Math Practice**

**Apply Mathematics**

Finding the number of occurrences of 2 heads is the same as finding the number of ways that you can choose 2 out of $n$ coins to show heads. What expression can you use to represent this number?

c. Complete the table showing the numbers of ways in which 2 heads can occur when $n$ coins are flipped.

| $n$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| **Occurrences of 2 heads** | | | | | |

d. Describe the pattern shown in part (c). Use the pattern to find the number of ways in which 2 heads can occur when 7 coins are flipped. Justify your answer.

## Probability Distributions

A **random variable** is a variable whose value is determined by the outcomes of a probability experiment. For example, when you roll a six-sided die, you can define a random variable *x* that represents the number showing on the die. So, the possible values of *x* are 1, 2, 3, 4, 5, and 6. For every random variable, a *probability distribution* can be defined.

### KEY IDEA

#### Probability Distributions

A **probability distribution** is a function that gives the probability of each possible value of a random variable. The sum of all the probabilities in a probability distribution must equal 1.

| Probability Distribution for Rolling a Six-Sided Die | | | | | | |
|---|---|---|---|---|---|---|
| **x** | 1 | 2 | 3 | 4 | 5 | 6 |
| **P(x)** | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

---

**EXAMPLE 1**   **Constructing a Probability Distribution**   ▶ WATCH

Let *x* be a random variable that represents the sum when two six-sided dice are rolled. Make a table and draw a histogram showing the probability distribution for *x*.

#### SOLUTION

**STUDY TIP**

Recall that there are 36 possible outcomes when rolling two six-sided dice. These are listed in Example 3 on page 406.

**Step 1**  Make a table. The possible values of *x* are the integers from 2 to 12. The table shows how many outcomes of rolling two dice produce each value of *x*. Divide the number of outcomes for *x* by 36 to find *P(x)*.

| x (sum) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Number of Outcomes** | 1 | 2 | 3 | 4 | 5 | 6 | 5 | 4 | 3 | 2 | 1 |
| **P(x)** | $\frac{1}{36}$ | $\frac{1}{18}$ | $\frac{1}{12}$ | $\frac{1}{9}$ | $\frac{5}{36}$ | $\frac{1}{6}$ | $\frac{5}{36}$ | $\frac{1}{9}$ | $\frac{1}{12}$ | $\frac{1}{18}$ | $\frac{1}{36}$ |

**Step 2**  Draw a histogram where the intervals are given by *x* and the relative frequencies are given by *P(x)*.
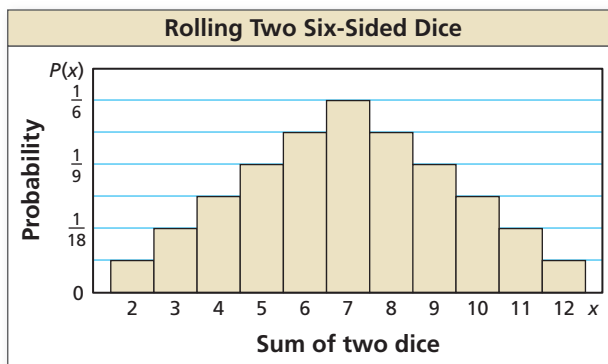
**EXAMPLE 2** **Interpreting a Probability Distribution**

Use the probability distribution in Example 1 to answer each question.

**a.** What is the most likely sum when rolling two six-sided dice?

**b.** What is the probability that the sum of the two dice is at least 10?

### SOLUTION

**a.** The most likely sum when rolling two six-sided dice is the value of $x$ for which $P(x)$ is greatest. This probability is greatest for $x = 7$. So, when rolling the two dice, the most likely sum is 7.

**b.** The probability that the sum of the two dice is at least 10 is

$$P(x \geq 10) = P(x = 10) + P(x = 11) + P(x = 12)$$

$$= \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

$$\approx 0.167.$$

▶ The probability is about 16.7%.

**SELF-ASSESSMENT** 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

An octahedral die has eight sides numbered 1 through 8. Let $x$ be a random variable that represents the sum when two such dice are rolled.

**1.** Make a table and draw a histogram showing the probability distribution for $x$.

**2.** What is the most likely sum when rolling the two dice?

**3.** What is the probability that the sum of the two dice is at most 3?

**4.** **WRITING** Explain why the sum of all the probabilities in a probability distribution must equal 1.

## Binomial Distributions

One type of probability distribution is a **binomial distribution**. A binomial distribution shows the probabilities of the outcomes of a *binomial experiment*.

### KEY IDEA

**Binomial Experiments**

A **binomial experiment** meets the following conditions.

• There are $n$ independent trials.

• Each trial has only two possible outcomes: success and failure.

• The probability of success is the same for each trial. This probability is denoted by $p$. The probability of failure is $1 - p$.

For a binomial experiment, the probability of exactly $k$ successes in $n$ trials is

$$P(k \text{ successes}) = {}_nC_k p^k(1 - p)^{n-k}.$$

**EXAMPLE 3** **Constructing a Binomial Distribution**

According to a survey, about 60% of teenagers ages 13 to 17 in the U.S. say they spend time with friends online daily. You ask 6 randomly chosen teenagers (ages 13 to 17) whether they spend time with friends online daily. Draw a histogram of the binomial distribution for your survey.

**SOLUTION**

The probability that a randomly selected teenager says they spend time with friends online daily is $p = 0.6$. Because you survey 6 people, $n = 6$.

**Math Practice**

**Calculate Accurately**
When probabilities are rounded, the sum of the probabilities may differ slightly from 1.

$P(k = 0) = {}_6C_0(0.6)^0(0.4)^6 \approx 0.004$

$P(k = 1) = {}_6C_1(0.6)^1(0.4)^5 \approx 0.037$

$P(k = 2) = {}_6C_2(0.6)^2(0.4)^4 \approx 0.138$

$P(k = 3) = {}_6C_3(0.6)^3(0.4)^3 \approx 0.276$

$P(k = 4) = {}_6C_4(0.6)^4(0.4)^2 \approx 0.311$

$P(k = 5) = {}_6C_5(0.6)^5(0.4)^1 \approx 0.187$

$P(k = 6) = {}_6C_6(0.6)^6(0.4)^0 \approx 0.047$

**Binomial Distribution for Your Survey**



A histogram of the distribution is shown.

**EXAMPLE 4** **Interpreting a Binomial Distribution**

Use the binomial distribution in Example 3 to answer each question.

**a.** What is the most likely outcome of the survey?

**b.** What is the probability that at most 2 teenagers spend time with friends online daily?

**COMMON ERROR**

Be sure you include $P(k = 0)$ when finding the probability that at most 2 teenagers spend time with friends online daily.

**SOLUTION**

**a.** The most likely outcome of the survey is the value of $k$ for which $P(k)$ is greatest. This probability is greatest for $k = 4$. The most likely outcome is that 4 of the 6 teenagers spend time with friends online daily.

**b.** The probability that at most 2 teenagers spend time with friends online daily is

$$P(k \leq 2) = P(k = 0) + P(k = 1) + P(k = 2)$$

$$\approx 0.004 + 0.037 + 0.138$$

$$= 0.179.$$

▶ The probability is about 17.9%.

**SELF-ASSESSMENT**  [1] I do not understand.  [2] I can do it with help.  [3] I can do it on my own.  [4] I can teach someone else.

According to a survey, about 26% of people ages 12 and older in the U.S. have listened to a podcast in the last month. You ask 4 randomly chosen people ages 12 and older whether they have listened to a podcast in the last month.

5. Draw a histogram of the binomial distribution for your survey.

6. What is the most likely outcome of your survey?

7. What is the probability that at most 2 people you survey have listened to a podcast in the last month?

**In Exercises 1–4, make a table and draw a histogram showing the probability distribution for the random variable.** ▶ *Example 1*

**1.** $x$ = the number on a table tennis ball randomly chosen from a bag that contains 5 balls labeled "1," 3 balls labeled "2," and 2 balls labeled "3."

**2.** $c = 1$ when a randomly chosen card out of a standard deck of 52 playing cards is a heart, $c = 2$ when it is a diamond, and $c = 3$ otherwise.

**3.** $w = 1$ when a randomly chosen letter from the English alphabet is a vowel and $w = 2$ otherwise.

**4.** $n$ = the number of digits in a random integer from 0 through 999.

**In Exercises 5 and 6, use the probability distribution to determine (a) the number that is most likely to be spun on a spinner, and (b) the probability of spinning an even number.** ▶ *Example 2*

**5.**



Spinner Results

**6.**



Spinner Results
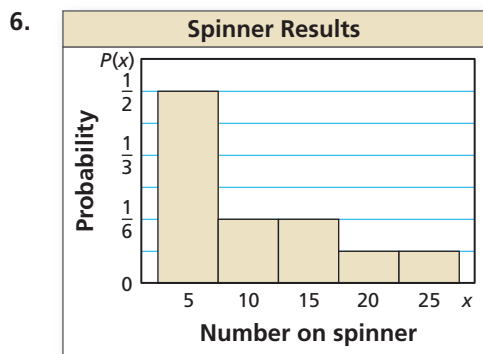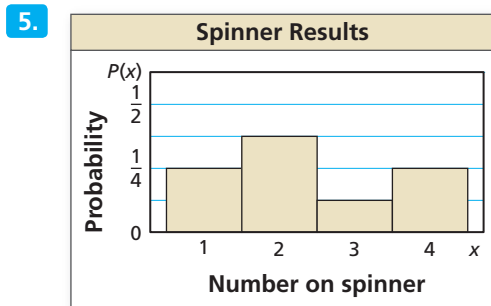
**USING EQUATIONS  In Exercises 7–10, calculate the probability of flipping a coin 20 times and getting the given number of heads.**

**7.** 1

**8.** 4

**9.** 18

**10.** 20

**11.  MODELING REAL LIFE**  In your school, 30% of students plan to attend a movie night. You ask 5 randomly chosen students from your school whether they plan to attend the movie night. ▶ *Examples 3 and 4*

  **a.** Draw a histogram of the binomial distribution for your survey.

  **b.** What is the most likely outcome of your survey?

  **c.** What is the probability that at most 2 students plan to attend the movie night?

**12.  MODELING REAL LIFE**  In your school, 70% of students have completed a required service project. You ask 8 randomly chosen students in your school whether they have completed the service project.



  **a.** Draw a histogram of the binomial distribution for your survey.

  **b.** What is the most likely outcome of your survey?

  **c.** What is the probability that at most 3 students have completed the service project?

**13.  ERROR ANALYSIS**  Describe and correct the error in finding the probability of rolling a 1 exactly 3 times in 5 rolls of a six-sided die.

$$P(k = 3) = {}_5C_3\left(\frac{1}{6}\right)^{5-3}\left(\frac{5}{6}\right)^3$$
$$\approx 0.0161$$

**14.  ERROR ANALYSIS**  Describe and correct the error in finding the probability of rolling a 2 or 4 exactly 1 time in 6 rolls of a six-sided die.

$$P(k = 1) = {}_6C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{6-1}$$
$$\approx 0.4019$$

**15.** **MP** **PROBLEM SOLVING** A sound system has $n$ speakers. Each speaker functions with probability $p$, independent of the other speakers. The system will function when at least 50% of its speakers function. For what values of $p$ is a 5-speaker system more likely to function than a 3-speaker system?

**16.** **HOW DO YOU SEE IT?**
The results of a binomial experiment with six trials are shown, where $P(x)$ represents the probability of $x$ successes. What is the least likely outcome of the experiment? Explain.

**Experiment Results**
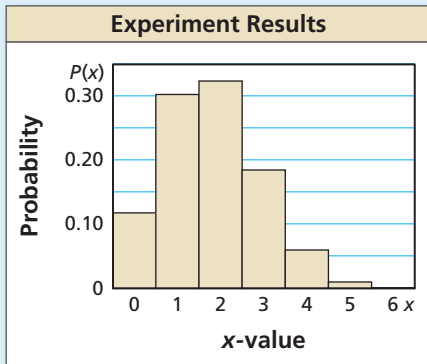


**17.** **MP** **REASONING** The probability of five successes in a binomial experiment with five trials is 1.024%. What is the probability of success for any individual trial?

**18.** **THOUGHT PROVOKING**
How many successes might you expect from a binomial experiment with $n$ trials and probability of success $p$? (*Hint*: Find the mean number of successes for $n$ trials.)

**19.** **CONNECTING CONCEPTS** On the farm shown, 7 gopher holes appear each week. Assume that a gopher hole has an equal chance of appearing at any point on the farm. What is the probability that at least one gopher hole appears in the carrot patch?



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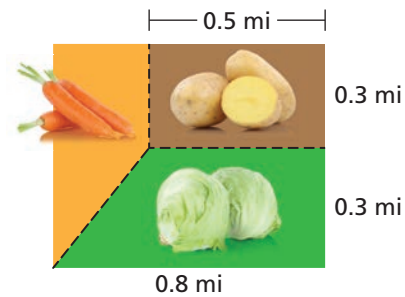
## REVIEW & REFRESH

**20.** Use the probability distribution below to determine the most likely number of weekdays with precipitation for one week.

**Precipitation Predictions**
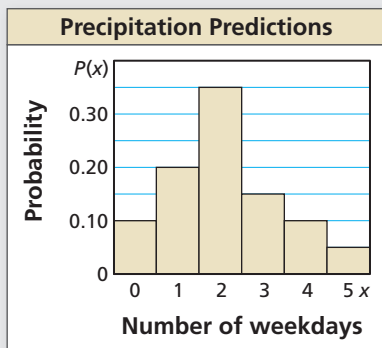


**In Exercises 21 and 22, count the possible combinations of $r$ letters chosen from the given list.**

**21.** E, F, G, H; $r = 2$

**22.** L, M, N, O, P; $r = 3$

**In Exercises 23 and 24, divide.**

**23.** $(x^2 - 5x - 24) \div (x - 8)$

**24.** $(x^4 + 6x^3 + 3x^2 - 26x - 24) \div (x + 2)$

**25.** Let the graph of $g$ be a translation 2 units right, followed by a horizontal shrink by a factor of $\frac{1}{2}$ of the graph of $f(x) = x^2 - x$. Write a rule for $g$.

**26.** **MODELING REAL LIFE** You collect data about a dog pageant. Of the 25 dogs in the pageant, 7 receive a ribbon, 18 receive a collar, and 5 receive both a ribbon and a collar. What is the probability that a dog in the pageant receives a ribbon *or* a collar?

**27.** A bag contains three $10 gift cards, two $20 gift cards, and a $30 gift card. You randomly select a gift card and give it away. Then you randomly select another gift card.

**Event $A$:** You select the $10 gift card first.

**Event $B$:** You select the $20 gift card second.

Tell whether the events are independent or dependent. Explain your reasoning.

**28.** Determine whether the graph represents an *arithmetic sequence*, a *geometric sequence*, or *neither*. Explain your reasoning.
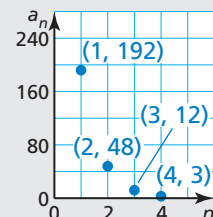
# 8 Chapter Review WITH CalcChat®

**Chapter Learning Target**     Understand probability.

**Chapter Success Criteria**
- ◆ I can define theoretical and experimental probability.
- ◆ I can use two-way tables to find probabilities.
- ■ I can compare independent and dependent events.
- ■ I can construct and interpret probability and        ◆ Surface
  binomial distributions.                                 ■ Deep

**SELF-ASSESSMENT**  | **1** I do not understand. | **2** I can do it with help. | **3** I can do it on my own. | **4** I can teach someone else. |
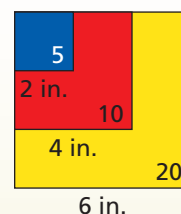
## 8.1   Sample Spaces and Probability   *(pp. 403–410)*   ▶ WATCH

**Learning Target:**  Find sample spaces and probabilities of events.

1. You flip a coin and draw a marble at random from a bag containing two blue marbles and two green marbles. Find the number of possible outcomes in the sample space. Then list the possible outcomes.

2. A bag contains 9 tiles, one for each letter in the word HAPPINESS. You choose a tile at random. What is the probability that you choose a tile with the letter S? What is the probability that you choose a tile with a letter other than P?

3. You throw a dart at the board shown. Your dart is equally likely to hit any point inside the square board. Are you most likely to get 5 points, 10 points, or 20 points?

> 5 | 2 in.
> 10 | 4 in.
> 20 | 6 in.

**Vocabulary** AZ VOCAB

probability experiment
outcome
event
sample space
probability of an
   event
theoretical probability
geometric probability
experimental
   probability

## 8.2   Two-Way Tables and Probability   *(pp. 411–416)*   ▶ WATCH

**Learning Target:**  Use two-way tables to represent data and find probabilities.

4. A survey asks residents of the east and west sides of a city whether they support the construction of a mall. The results, given as joint relative frequencies, are shown in the two-way table. What is the probability that a randomly selected resident who responded no is from the west side?

|  |  | Location | |
|---|---|---|---|
|  |  | **East Side** | **West Side** |
| **Response** | **Yes** | 0.47 | 0.36 |
|  | **No** | 0.08 | 0.09 |

5. After an assembly, 220 boys and 270 girls respond to a survey. Of those, 200 boys and 230 girls say the motivational speaker was impactful. Organize these results in a two-way table. Then find and interpret the marginal frequencies.

**Vocabulary** AZ VOCAB

two-way table
joint frequency
marginal frequency
joint relative
   frequency
marginal relative
   frequency
conditional relative
   frequency

## 8.3 Conditional Probability *(pp. 417–424)* ▶ WATCH

**Learning Target:** Find and use conditional probabilities.

**Vocabulary** [AZ VOCAB]

conditional probability

6. You have two acrylic paintings and two oil paintings. You randomly select a painting to sell and then you randomly select a different painting to present at a gallery. Find the probability that you selected an acrylic painting to sell given that you randomly selected an oil painting to present at the gallery.

7. An inspector tests rotors of helicopter drones. The two-way table shows the joint and marginal relative frequencies. Find and compare $P(\text{single rotor} \mid \text{failed})$ and $P(\text{single rotor} \mid \text{passed})$.

|      |              | Result |        |       |
|------|--------------|--------|--------|-------|
|      |              | Passed | Failed | Total |
| Type | Single Rotor | 0.30   | 0.20   | 0.50  |
|      | Multi-Rotor  | 0.40   | 0.10   | 0.50  |
|      | Total        | 0.70   | 0.30   | 1     |

8. At a baseball game, 90% of guests receive a coupon code and 30% of guests receive a coupon code and a bobblehead. What is the probability that a guest who receives a coupon code also receives a bobblehead?

## 8.4 Independent and Dependent Events *(pp. 425–432)* ▶ WATCH

**Learning Target:** Understand and find probabilities of independent and dependent events.

**Vocabulary** [AZ VOCAB]

independent events
dependent events

9. As part of a board game, you need to spin the spinner, which is divided into equal parts. Find the probability that you get a 2 on your first spin and a number less than or equal to 5 on your second spin.

10. You are a DJ at a wedding. A playlist contains 10 pop songs and 20 country songs. You set the playlist to select songs at random. Once a song is played, the same song will not play again. Find the probability that the first two songs to play are both country songs.

## 8.5 Probability of Disjoint and Overlapping Events *(pp. 433–438)* ▶ WATCH

**Learning Target:** Find probabilities of disjoint and overlapping events.

**Vocabulary** [AZ VOCAB]

compound event
overlapping events
disjoint events

11. Let $A$ and $B$ be events such that $P(A) = 0.32$, $P(B) = 0.48$, and $P(A \text{ and } B) = 0.12$. Find $P(A \text{ or } B)$.

12. Out of 100 employees in a restaurant, 92 either work part time or work 5 days each week. There are 14 employees who work part time and 80 employees who work 5 days each week. What is the probability that a randomly selected employee works both part time and 5 days each week?

**Permutations and Combinations** *(pp. 439–448)* ▶ WATCH

GO DIGITAL

**Learning Target:** Count permutations and combinations.

**Vocabulary** AZ VOCAB

permutation
*n* factorial
combination
Binomial Theorem

**Evaluate the expression.**

**13.** $_7P_6$

**14.** $_{13}P_{10}$

**15.** $_6C_2$

**16.** $_8C_4$

**17.** You and your friend are two of the four winners of individual concert tickets. There is one VIP ticket, one superior ticket, one general admission ticket, and one value ticket. The tickets are given to the winners randomly. Find the probability that you get the superior ticket and your friend gets the value ticket.

**18.** You work in a food truck at a festival. Of the 11 food trucks, 2 are randomly selected to be placed at the entrance. What is the probability that your food truck is placed at the entrance?

**Use the Binomial Theorem to write the expansion of the expression.**

**19.** $(3x - 2y)^5$

**20.** $(2x + y^2)^4$

**Binomial Distributions** *(pp. 449–454)* ▶ WATCH

**Learning Target:** Understand binomial distributions.

**Vocabulary** AZ VOCAB

random variable
probability
  distribution
binomial distribution
binomial experiment

**21.** Find the probability of flipping a coin 12 times and getting exactly 4 heads.

**22.** A basketball player makes a free throw 82.6% of the time. The player attempts 5 free throws. Draw a histogram of the binomial distribution of the number of successful free throws. What is the most likely outcome?

**23.** According to a survey, about 37% of Americans go online mostly using a smartphone. You ask 4 randomly chosen Americans about whether they go online mostly using a smartphone. Draw a histogram of the binomial distribution for your survey. What is the probability that two or more of the respondents go online mostly using a smartphone?

## Mathematical Practices

### Model with Mathematics

*Mathematically proficient students apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

**1.** Describe a real-life situation that can be represented by a binomial distribution.

**2.** In Exercise 27 on page 410, explain how you used probability to determine the amount that you will charge and the amount that each winner will receive.

**3.** In Exercise 10 on page 422, what types of decisions can the owner of the gas station make using the given probabilities?

1. You randomly choose one meat and one dressing for a salad. How many possible outcomes are in the sample space? List the possible outcomes.

| • MEAT • | • DRESSING • |
|---|---|
| Chicken | Ranch |
| Steak | Italian |
| Salmon | French |

**Evaluate the expression.**

2. $_7P_2$

3. $_6C_3$

4. Use the Binomial Theorem to write the binomial expansion of $(x + y^2)^5$.

5. According to a survey, about 54% of teenagers in the U.S. say they spend too much time on their cell phones. You ask 5 randomly chosen teenagers whether they think they spend too much time on their cell phones.

   a. Draw a histogram of the binomial distribution for the survey.

   b. What is the most likely outcome of the survey?

   c. What is the probability that at least 3 teenagers you survey say they spend too much time on their cell phones?

6. A bag contains the gift cards shown in the table.

   a. You randomly draw a gift card. Find the probability that you draw a $25 mall gift card.

   b. You randomly draw a gift card. Find the probability that you draw a $50 gift card or a restaurant gift card.

   c. You randomly draw a gift card, set it aside, and then draw a second gift card for your friend. Find the probability that you draw a mall gift card, then a gasoline gift card.

|  | Mall | Gasoline | Restaurant |
|---|---|---|---|
| **$25** | 4 | 2 | 3 |
| **$50** | 1 | 3 | 2 |

7. Describe why it is necessary to subtract $P(A$ and $B)$ when finding $P(A$ or $B)$ for two overlapping events. Then describe why it is *not* necessary to subtract $P(A$ and $B)$ when finding $P(A$ or $B)$ for two disjoint events.

8. You roll a six-sided die twice. Find the probability that the first number is in the domain of $f(x) = \sqrt{x - 2}$ and the second number is in the domain of $g(x) = \sqrt{x - 5}$.

9. You are choosing a new cell phone carrier. The three plans you consider are equally priced. You ask several of your neighbors whether they are satisfied with their current cell phone carrier. The table shows the results. According to this survey, which carrier should you choose?

| Carrier | Satisfied | Not Satisfied |
|---|---|---|
| A | IIII | II |
| B | IIII | III |
| C | IIII I | IIII |

10. Three volunteers are chosen at random from a group of 12 to help at a camp.

    a. What is the probability that you, your brother, and your friend are chosen?

    b. The first person chosen will be a counselor, the second will be a lifeguard, and the third will be a cook. What is the probability that you are the cook, your brother is the lifeguard, and your friend is the counselor?

# 8 Performance Task
# Buried Treasures

Archaeologists study a wide variety of artifacts in order to learn about the past.

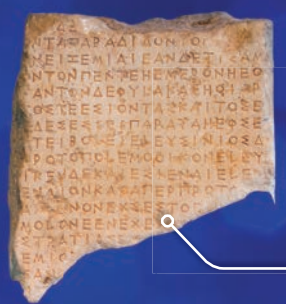### COINS
Coins are often made with words and images that provide valuable information about nations and rulers.

### FOSSILS
Fossils provide information about plant and animal species from different time periods.

### POTTERY
Pottery, one of the most common types of archaeological discovery, gives cultural insight into ancient societies.

### INSCRIPTIONS
Inscriptions and manuscripts give insight into culture, language, chronology, and much more.

### BONES
Bones provide information about topics such as lifestyle, health, and ages of civilizations.
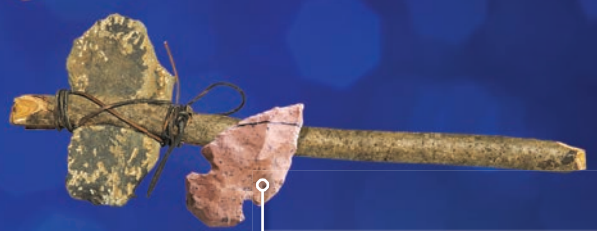
### TOOLS
Ancient tools demonstrate the technologies and behaviors of ancient peoples.

## CHOOSING AN EXCAVATION SITE

**INFO**

Archaeologists will choose one of three sites to excavate. Multiple surveys are used to determine whether each site appears likely to contain a significant number of ancient artifacts. The conclusions of the surveys are shown. Which site should the archaeologists choose? Create a presentation to convince the archaeologists to choose this site.

The site is partitioned into 9 square-shaped regions. In any given region, the probability of finding an artifact is about 45%. Draw and interpret a histogram of a probability distribution for this situation.

| Site | Likely | Unlikely |
|------|--------|----------|
| A | 𝍸𝍸I | 𝍸 |
| B | 𝍸III | III |
| C | 𝍸IIII | IIII |

▶ WATCH  Tutorial videos are available for each exercise.

**1.** Which statements describe the transformation of the graph of $f(x) = x^3 - x$ represented by $g(x) = 4(x - 2)^3 - 4(x - 2)$?

Ⓐ a vertical stretch by a factor of 4

Ⓑ a vertical shrink by a factor of $\frac{1}{4}$

Ⓒ a horizontal shrink by a factor of $\frac{1}{4}$

Ⓓ a horizontal stretch by a factor of 4

Ⓔ a horizontal translation 2 units right

Ⓕ a horizontal translation 2 units left

**2.** You randomly select a size and flavor for a shake from the menu shown. How many possible outcomes are in the sample space? List the possible outcomes. Then find the probability that the selected shake is *not* a peanut butter shake.



*Sizes*
• Small
• Medium
• Large

SPECIALTY

*Flavors*
• Chocolate
• Vanilla
• Strawberry
• Banana
• Peanut Butter

SHAKES

**3.** Which of the following is the condensed form of the logarithmic expression $3 \ln x - \ln 7$?

Ⓐ $\ln(x^3 - 7)$         Ⓑ $\ln(3x - 7)$

Ⓒ $\ln \dfrac{3x}{7}$         Ⓓ $\ln \dfrac{x^3}{7}$

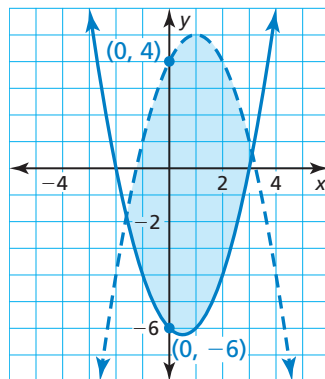**4.** The graphs of $f(x) = \dfrac{x + 1}{x + 2}$ and $g(x) = 2$ are shown. What is the solution of $f(x) = g(x)$?
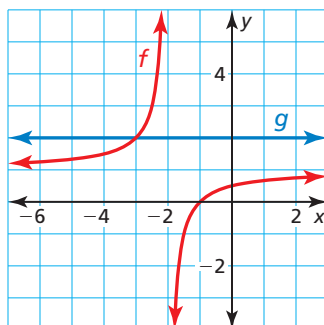
5. When you divide $2x^3 + x^2 - 5x + k$ by $x^2 - 3x + 1$, the remainder is $14x + 13$. What is the value of $k$?

6. A survey asked male and female students about whether they prefer to take gym class or choir. The table shows the results.

| | | Class | | |
| --- | --- | --- | --- | --- |
| | | Gym | Choir | Total |
| Gender | Male | | | 50 |
| | Female | 23 | | |
| | Total | | 49 | 106 |

   a. Complete the two-way table.

   b. What is the probability that a randomly selected student from the survey is female and prefers choir?

   c. What is the probability that a randomly selected male student from the survey prefers gym class?

7. The graph of which system of inequalities is shown?



   Ⓐ $y \le -(x-1)^2 + 5$
   $y > (x+2)(x-3)$

   Ⓑ $y > -(x-1)^2 + 5$
   $y \ge (x+2)(x-3)$

   Ⓒ $y < -(x-1)^2 + 5$
   $y \ge (x+2)(x-3)$

   Ⓓ $y < -(x-1)^2 + 5$
   $y \le (x+2)(x-3)$

8. A card is randomly selected from a standard deck of 52 playing cards. What is the probability that it is a red card or a king?

   Ⓐ $\frac{1}{26}$

   Ⓑ $\frac{7}{13}$

   Ⓒ $\frac{15}{26}$

   Ⓓ $\frac{8}{13}$

9. Complete the equation.

$$\left(\boxed{\phantom{0}} - 3i\right) + \left(4 + \boxed{\phantom{0}}\, i\right) = 10 + 5i$$